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Bayesian Decision Making

Theory

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Chapter 1

Introduction

This text should support learning theory of Bayesian decision making under uncertainty presented at the course of “Adaptive Systems”. There are good textbooks for learning both decision making theory, e.g. [1, 2] and adaptive control, e.g. [3, 4]. The view presented in this text, namely, viewing adaptive systems as an important branch of *dynamic* decision making under uncertainty is dissipated in various textbooks and numerous papers. The classical text [5, 6] is probably closest to this one. Permanently growing computer power available, novel solutions to various theoretical and algorithmic subtasks justify this attempt to update the description of the theory.

1.1 Aim of the work

Theory of statistical decision making under uncertainty [2, 7] provides a unified logical structure for solving problems ranging from parameter estimation, prediction, pattern recognition, learning, testing of hypothesis etc. up to feedback control. The application width is also enormous: inspection, maintenance and control of technologic as well as economic processes; signal processing; information processing in medicine, physics, etc. Huge amount of available particular results, techniques and methods hide the common logical structure and leave often newcomers without a guide.

This text aims to recall the relatively simple common structure of the mentioned decision tasks and to help the user in focusing on those ingredients that are specific to his/her problem.

Naturally, the reaching of a satisfactory solution of a particular decision making problem remains to be an iterative process [8]. The number of trial solutions can be, however, significantly reduced by using the advocated logical structure. This is the main practical outcome of mastering the presented theory.

The logical structure of the decision making under uncertainty is now well understood. The number of open problems, related especially to computational and implementation aspects, is, however, large. Some of them are formulated throughout the text. In this way, we try to *turn attention of (prospective) researchers to challenging and important topics.*

1.2 Structure

The aim of the work dictates the top-down style of presentation: the logically simple but relatively abstract problem formulation and solution are gradually specialised. The logical skeleton of the theory is filled up by practical details later on in Chapters dealing with particular examples.

Chapter 2 summarises the theory of the design. Basic analytical results are in Chapter 3. Usual decision making tasks are discussed, formulated and solved in Chapter 4.

Chapter 2

Underlying theory

At a general level, the presented decision-making theory should help the decision maker – typically, human being – to select one of the available options (decisions). These options relate to a part of the real world, to a system. They concern with descriptions of the system and/or its influencing.

This chapter summarizes the design principles and tools exploited later on in our particular task, i.e. operator support. It starts with conventions and notions used throughout, Sections 2.1, 2.2. The considered framework covers a broad range of problems. Inevitably, the adopted symbols and notions have more specific meanings in specific application fields. The reader is kindly asked to be patient especially in this respect.

The adopted principle of the optimal decision-making under uncertainty, inspected in Section 2.3, implies the following important conclusion: incomplete knowledge and randomness have the same operational consequences for decision-making and should be treated in the same way labelled as *Bayesian decision making*. In the same section, the design of optimal decision rules is solved. In Section 2.4 the design of the optimal strategies is derived. These designs work with models that are usually obtained through Bayesian learning described in Section 2.5.

A similarly formulated control design, that is a specific decision-making problem, is presented in [9]. For a detailed explanation of Bayesian learning see [10]. This chapter is a tailored version of a chapter in [11] where our view on decision making is summarized.

2.1 General conventions

The conventions presented here are mostly respected in this work. If some exception is necessary it is named just at the place of its validity. If some verbal notions are introduced within bodies of Propositions, Remarks etc., then they are *emphasized by the print* that differs from that of the surrounding text. Moreover, they appear in Index. Sometimes, important parts of sentences are stressed by underlining them.

f is the letter reserved for probability (density) functions ($p(d)f$).

The meaning of the $p(d)f$ is given through the name of its argument.

x^* denotes the range of x , $x \in x^*$.

\dot{x} denotes the number of members in the countable set x^* or number of entries in the vector x .

\equiv means the equality by definition.

x_t is a quantity x at the discrete time labelled by $t \in t^* \equiv \{1, \dots, \mathring{t}\}$.

$\mathring{t} \leq \infty$ is called (decision, learning, prediction, control, advising) horizon.

$x_{i;t}$ is an i -th entry of the array x at time t .

The semicolon in the subscript indicates that the symbol following is time index.

$x(k_l)$ denotes the sequence x_t with t between time moments $k \leq l$, i.e. $x(k_l) \equiv x_k, \dots, x_l$.

$x(t) \equiv x(1_t)$.

$x(t)$ is an empty sequence and reflects just the prior information if $t < 1$.

$\text{supp}[f(x)]$ is the support of the pdf $f : x^* \rightarrow [0, \infty]$, i.e. the subset of x^* on which $f(x) > 0$.

2.2 Basic notions and notations

Below, if need be, a brief characterization of the introduced notion is complemented by explanatory comments (printed differently).

Quantity is a mapping with a numerical range, i.e. a subset of the multi-variate, real-valued space.

The domain and form of the quantity are mostly unused and unspecified. The introduced notion corresponds with *random variable* used in probability theory. The use of the alternative term should stress that probability is not our primary aim but the tool we adopted for our main goal, i.e. decision making under uncertainty. Moreover, we stress our orientation on numerical values that arise mostly by measuring physical quantities.

Realization is a value of the quantity for its fixed argument.

Often, the quantity and its realization are not distinguished, as usual. The proper meaning is determined by the context.

System is the part of the world that is of interest for a decision maker who should either describe or influence it.

Decision maker might be a person, group of persons or mechanisms.

The system is specified with respect to the aim that the decision maker wants to reach and with respect to the tools he/she has available. In other words, the boundaries (that can be penetrated) of the system are implied by the decision task.

Decision $a \in a^*$ is the value of a quantity that can be directly chosen by the decision maker for reaching his/her aims.

A decision task makes sense only when the decision maker faces real option, i.e. when $\mathring{a} > 1$.

(Decision) experience $\mathcal{P}_a \in \mathcal{P}_a^*$ is knowledge about the system available to the decision maker for the selecting the decision $a \in a^*$.

For example, if just data values D are available for constructing the estimate $\hat{\Theta}$ of an unknown quantity $\Theta \in \Theta^*$ then the experience is $\mathcal{P}_{\hat{\Theta}} \equiv D$. Often, experience includes the past data observed.

Note that the symbol \mathcal{P}_{a^*} would be more adequate than \mathcal{P}_a . Its use would, however, made the notation even more cumbersome.

(Decision) ignorance $\mathcal{F}_a \in \mathcal{F}_a^*$ is knowledge about the system unavailable to the decision maker for the choice of the value of a .

The unknown value of the quantity Θ to be estimated is a part of the ignorance $\mathcal{F}_{\hat{\Theta}}$ of its estimate $\hat{\Theta}$. Often, ignorance contains future, still unobserved data.

(System) behavior \mathcal{Q}^* consists of all possible realizations (of trajectories) \mathcal{Q} , i.e. values of all quantities within the time span determined by the horizon of interest that are related to the system and considered by the decision maker.

The realization \mathcal{Q} can be split with respect to any decision $a \in a^*$ into the relevant experience \mathcal{P}_a and ignorance \mathcal{F}_a . Formally, $\mathcal{Q} = (\mathcal{P}_a, a, \mathcal{F}_a)$.

Note that a single realization \mathcal{Q} can be split differently with respect to a pair of decisions, say a, \tilde{a} with different experience $\mathcal{P}_a \neq \mathcal{P}_{\tilde{a}}$ and different ignorance $\mathcal{F}_a \neq \mathcal{F}_{\tilde{a}}$. $\mathcal{Q} = (\mathcal{P}_a, a, \mathcal{F}_a) = (\mathcal{P}_{\tilde{a}}, \tilde{a}, \mathcal{F}_{\tilde{a}})$.

(System) input $u \in u^*$ is a decision, which is supposed to influence the ignorance part \mathcal{F}_u of the (system) behavior.

A manipulated valve position influencing a fluid flow is the system input. A point estimate $\hat{\Theta}$ of an unknown (realization of) quantity Θ is an instance of the decision that is not the system input. It is chosen with the aim to describe the system and exhibits no direct influence on it.

(System) output $y \in y^*$ is an observable quantity that provides the decision maker information about the (system) behavior.

A pressure measured in an isolated heated system is an instance of the output. A pressure applied on a closed system is an instance of the input.

To be or not to be input or output is a relative property.

Innovation $\Delta_{a_t} \in \Delta_{a_t}^*$ contains quantities that are not included in the experience \mathcal{P}_{a_t} but are included in $\mathcal{P}_{a_{t+1}} \setminus a_t$.

Often, $\Delta_{a_t} = y_t =$ the system output at time t .

Decision rule $\mathcal{R} : \mathcal{Q}^* \rightarrow a^*$ is a mapping that assigns a decision $a \in a^*$ to the behavior $\mathcal{Q} \in \mathcal{Q}^*$.

Causal decision rule $\mathcal{R} : \mathcal{P}_a^* \rightarrow a^*$ is a mapping that assigns the decision $a \in a^*$ to its experience $\mathcal{P}_a \in \mathcal{P}_a^*$.

In other words, the decision a made by a causal decision rule is uninfluenced by the related ignorance \mathcal{F}_a .

Mostly, we deal with the causal decision rules so that the term “causal” is often dropped.

Estimator is an instance of the causal decision rule $\mathcal{R} : \mathcal{P}_{\hat{\Theta}}^* \rightarrow \hat{\Theta}^*$ that assigns an estimate $\hat{\Theta}$ of the unknown quantity $\Theta \in \Theta^*$ to the available experience $\mathcal{P}_{\hat{\Theta}}$.

Strategy is a sequence of decision rules $\{\mathcal{R}_t : \mathcal{Q}^* \rightarrow a_t^*\}_{t \in t^*}$.

Causal strategy $\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}$ is a sequence made of causal decision rules.

Again, we deal mostly with causal strategies so that the term “causal” is often dropped.

Controller is a causal strategy assigning inputs u_t to experience \mathcal{P}_{u_t} , $\forall t \in t^*$.

The proportional controller given by the constant p is an example of the causal control strategy $\{y_{t-1}^* \rightarrow u_t^* : u_t = py_{t-1}\}_{t \in t^*}$.

Design selects the decision rule or strategy.

The design selecting a single rule is called *static design*. The choice of the strategy is called *dynamic design*.

The person (group) who makes the selection is *designer*. Authors and readers of this text are supposed to be designers. In that sense, the term *we* used within the text should mostly be read: *we designers*.

Agreement 2.2.0.1 (Interface between the design and reality) *Physical connections of the design domain to the real world (sensors, transmission lines, actuators etc.) are here always taken as a part of the system. Consequently, all quantities and mappings considered can be and are taken as mathematical entities living in an abstract calculating machine.*

Uncertain behavior (related to static design) is a behavior whose realizations \mathcal{Q} can be decomposed into

- $\mathcal{Q}_{\mathcal{R}} \equiv$ the part that is unambiguously determined by the considered decision rule $\mathcal{R} \in \mathcal{R}^*$,
- uncertainty $\Upsilon \equiv$ the part of the behavior that belongs to the ignorance $\mathcal{F}_{\mathcal{R}(\mathcal{P})}$ of decisions $\mathcal{R}(\mathcal{P})$ generated by the admissible rules $\mathcal{R} \in \mathcal{R}^*$ and uninfluenced by them, even indirectly.

With an abuse of notation, we write the corresponding decomposition of realization $\mathcal{Q} = (\mathcal{Q}_{\mathcal{R}}, \Upsilon) = (\cdot, \text{uncertainty})$.

Uncertain behavior related to dynamic design is encountered if any of its rules faces uncertainty.

Incomplete knowledge of (the realization of) a considered quantity $\Theta \in \Theta^*$ makes the behavior uncertain.

External unobserved noise influencing the system makes its behavior uncertain.

Uncertainty expresses both incomplete knowledge and randomness.

Uncertain behavior in the dynamic design is defined in a similar way by replacing rules with strategies. The related cumbersome notation is avoided in our presentation.

Decision-making means design and application of a decision rule (strategy).

Admissible strategy is a strategy $\{\mathcal{R}_t\}_{t \in t^*}$ that

- is causal $\Leftrightarrow \{\mathcal{R}_t\}_{t \in t^*} \equiv \{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}$ and

- meets *technological restrictions* \Leftrightarrow the ranges of its decision rules are in pre-specified subsets of respective decision sets.

Loss function, $\mathcal{Z} : \mathcal{Q}^* \rightarrow [0, \infty]$, quantifies the degree of achievement of the design aim.

The loss function measures quality of the realizations \mathcal{Q} and indirectly (partially) orders admissible strategies influencing the behavior.

The smaller the value of $\mathcal{Z}(\mathcal{Q})$ the better.

Negative value of profit might be a loss function in economic applications.

Quadratic deviation $(\hat{\Theta} - \Theta)'(\hat{\Theta} - \Theta)$ (' is transposition) of the estimate $\hat{\Theta}$ and of an unknown vector Θ is a wide spread loss function used for ranking decision rules generating point estimates.

The important case of multi-valued loss function [12] is out of the scope of this text but the approach discussed in Section 2.3 could be relatively simply extended to it by embedding the index of the “decisive” entry of the loss function into uncertainty.

“Expected” loss $\tilde{\mathcal{E}}(\mathcal{Z}) \equiv \tilde{\mathcal{E}}_{\{\mathcal{R}\}}(\mathcal{Z})$ assigns to the considered loss function \mathcal{Z} and strategy $\{\mathcal{R}\}$ a value in $[0, \infty]$ that is independent of the uncertainty.

The quotation marks as well as the sign $\tilde{}$ serve us in the discussion that shows that, under widely acceptable conditions, we have to deal with expectation in mathematical sense. Then, they are unused any more.

Optimal design selects an admissible strategy that leads to the smallest value of the “expected” loss function.

Practically admissible strategy is an admissible strategy that respects restrictions limiting complexity of the decision-making.

The complexity is considered with respect to the computational resources available at the design and application stages. The majority of discussed problems in which the complexity restriction play a role are computationally hard in terms of computer sciences. An intuitive understanding of the computational complexity is sufficient to our purposes.

Practically optimal design selects a practically admissible strategy giving the smallest values of the “expected” loss.

The presented optimal design provides optimal admissible strategies and can be simply adapted to provide strategies of a pre-specified complexity by optimizing over a set of simple decision rules, for instance, over proportional controllers only.

Operational formal tools for practically optimal design are not available. It is not known how to make the optimal design of a pre-specified complexity.

We never know whether the selection of the constant determining proportional controller made with use of, say, ten algebraic operations is really the best one possible among all selections that are allowed to perform ten algebraic operations.

This is the *main barrier of the applicability* of the theory describing the optimal design. The optimal design becomes a practical tool by employing sound engineering heuristics. The practical optimum is not guaranteed. This fact is stressed by using the term *suboptimal design*.

2.3 Decision-making under uncertainty

Here, we describe a general way how to understand and face uncertainty that causes incomplete ordering of strategies. In order to avoid a cumbersome notation, we formulate the adopted design principle, related requirements and their consequences for the static design, i.e. the design of a single not necessarily causal decision rule. The obtained results are also valid for the dynamic design, i.e. the choice of decision strategies.

Agreement 2.3.0.2 (Uncertainty in decision-making) *Decision-making under uncertainty arises if the optimal decision-making is to be performed and*

- *at least a pair of different decisions can be made, $\hat{a} > 1$,*
- *the considered loss function $\mathcal{Z}(\mathcal{Q}) \equiv \mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon)$ depends on non-void set Υ^* of uncertainties.*

Under uncertainty, the function $\mathcal{Z}_{\mathcal{R}}(\Upsilon) \equiv \mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon)$ of uncertainty Υ , i.e. the part of realization in ignorance uninfluenced by the rule \mathcal{R} , is assigned to each considered decision rule $\mathcal{R} \in \mathcal{R}^$. The set of such functions is denoted $\mathcal{Z}_{\mathcal{R}^*}$*

$$\mathcal{Z}_{\mathcal{R}^*} \equiv \{\mathcal{Z}_{\mathcal{R}} : \Upsilon^* \rightarrow [0, \infty], \mathcal{Z}_{\mathcal{R}}(\Upsilon) \equiv \mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon)\}_{\mathcal{R} \in \mathcal{R}^*}. \quad (2.1)$$

Under uncertainty, the loss function is insufficient for a *complete ordering* (comparing) of admissible rules, and consequently strategies, in spite of the fact that its values are in the fully ordered set. Each \mathcal{R} is assigned a function in the set (2.1) and not a single number.

For instance, consider a pair of estimates $\hat{\Theta}_1 \neq \hat{\Theta}_2$ of an unknown scalar quantity $\Theta \in \Theta^* \equiv (-\infty, \infty)$. They cannot be compared with the help of the quadratic loss function $(\Theta - \hat{\Theta})^2$ as we do not know whether Θ is in that part of Θ^* where $(\Theta - \hat{\Theta}_1)^2 \leq (\Theta - \hat{\Theta}_2)^2$ or in its complement.

2.3.1 Ordering of decision rules

We rely on the following systematic choice of a good decision rule.

Agreement 2.3.1.1 (Design principle: “expectation”-minimization design)

- *A functional $\tilde{\mathcal{E}}_{\mathcal{R}}$, called “expectation”, is selected by designer. It assigns to functions in (2.1) (determined by the considered loss function \mathcal{Z} and indexed by inspected decision rules $\mathcal{R} \in \mathcal{R}^*$) a non-negative “expected loss” $\tilde{\mathcal{E}}_{\mathcal{R}}[\mathcal{Z}]$*

$$\begin{aligned} \tilde{\mathcal{E}}_{\mathcal{R}} &: \mathcal{Z}_{\mathcal{R}^*} \rightarrow [0, \infty] \\ \mathcal{Z}_{\mathcal{R}^*} &\equiv \{\mathcal{Z}_{\mathcal{R}} : \Upsilon^* \rightarrow [0, \infty], \mathcal{Z}_{\mathcal{R}}(\Upsilon) \equiv \mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon)\}_{\mathcal{R} \in \mathcal{R}^*} \end{aligned} \quad (2.2)$$

- *The minimizer of $\tilde{\mathcal{E}}_{\mathcal{R}}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon)] \equiv \tilde{\mathcal{E}}[\mathcal{Z}_{\mathcal{R}}]$ found in \mathcal{R}^* is taken as the optimal decision rule.*

The outcome of this design depends on the introduced “*expectation*” $\tilde{\mathcal{E}}_{\mathcal{R}}$. It has to avoid (at least) unequivocally bad rules. Such bad rules are identified here with dominated decision rules.

Agreement 2.3.1.2 (Dominated decision rules; strictly isotonic expectation) *Let a loss function \mathcal{Z} measure the quality of the behavior. The decision rule $\mathcal{R} : \mathcal{Q}^* \rightarrow a^*$ is called dominated iff there is another decision rule $\tilde{\mathcal{R}} : \mathcal{Q}^* \rightarrow a^*$ such that for*

$$\mathcal{Z}_{\mathcal{R}} \geq \mathcal{Z}_{\tilde{\mathcal{R}}} \Leftrightarrow \mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon) \geq \mathcal{Z}(\mathcal{Q}_{\tilde{\mathcal{R}}}, \Upsilon), \forall \Upsilon \in \Upsilon^*. \quad (2.3)$$

The dominated decision rule is called strictly dominated iff there is a non-trivial subset of Υ^ on which the inequality (2.3) is strict.*

The "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$ (2.2) is said to be strictly isotonic if for a decision rule \mathcal{R} strictly dominated by a decision rule $\tilde{\mathcal{R}}$ it holds

$$\tilde{\mathcal{E}}_{\mathcal{R}}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}}, \Upsilon)] > \tilde{\mathcal{E}}_{\tilde{\mathcal{R}}}[\mathcal{Z}(\mathcal{Q}_{\tilde{\mathcal{R}}}, \Upsilon)]$$

Now we can specify the dominated decision rules as those to be surely avoided.

Requirement 2.3.1.3 (Inadmissibility of strictly dominated decision rules) *The considered "expectation"-minimization design, Agreement 2.3.1.1, must not lead to the strictly dominated decision rule.*

We want to select the "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$ so that the decision rules are ordered as objectively as possible. It means that the ordering should weakly depend on the inspected set of rules \mathcal{R}^* , on the loss function \mathcal{Z} considered and on the "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$ selected by the designer.

Requirement 2.3.1.4 (Independence of \mathcal{R}^*) *The chosen "expectation" should fulfill Requirement 2.3.1.3 even if we restrict the set of possible decision rules \mathcal{R}^* to any of its non-trivial subsets that contains at least two different rules while at least one of them gives a finite "expected" loss.*

Proposition 2.3.1.5 (Isotonic ordering) *Assume that there is a rule in \mathcal{R}^* for which the "expected" loss is finite. Then, Requirements 2.3.1.3, 2.3.1.4 are fulfilled, i.e. the chosen rule is not dominated even if the set \mathcal{R}^* is restricted, iff the "expectation" is strictly isotonic, see Agreement 2.3.1.2.*

Proof:

1. We prove by contradiction that – with strictly isotonic "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$ – the minimizer cannot be strictly dominated.

Let $\tilde{\mathcal{E}}_{\mathcal{R}}[\mathcal{Z}]$ be strictly isotonic on its domain $\mathcal{Z}_{\mathcal{R}^*}$ (2.2) and $\mathcal{R}_o \in \mathcal{R}^*$ be a minimizer of the "expected" loss. The minimizer gives necessarily a finite value of the corresponding $\tilde{\mathcal{E}}_{\mathcal{R}}[\mathcal{Z}]$. Let $\mathcal{R}_d \in \mathcal{R}^*$ dominate it strictly. Then, because of the construction of \mathcal{R}_o , the strict dominance and strictly isotonic nature of $\tilde{\mathcal{E}}_{\mathcal{R}}$, we get the following contradictory inequality

$$\tilde{\mathcal{E}}_{\mathcal{R}_o}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}_o}, \Upsilon)] \underbrace{\leq}_{\text{minimum}} \tilde{\mathcal{E}}_{\mathcal{R}_d}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}_d}, \Upsilon)] \underbrace{\leq}_{\text{strictly isotonic}} \tilde{\mathcal{E}}_{\mathcal{R}_o}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}_o}, \Upsilon)].$$

2. We prove by contradiction that use of an "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$ that is not strictly isotonic leads to violation of Requirement 2.3.1.3 when Requirement 2.3.1.4 holds.

If $\tilde{\mathcal{E}}_{\mathcal{R}}[\mathcal{Z}]$ is not strictly isotonic on its domain $\mathcal{Z}_{\mathcal{R}^*}$ (2.2) then there is rule $\mathcal{R}_1 \in \mathcal{R}^*$ strictly dominated by a decision rule $\mathcal{R}_d \in \mathcal{R}^*$ such that

$$\tilde{\mathcal{E}}_{\mathcal{R}_d}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}_d}, \Upsilon)] \geq \tilde{\mathcal{E}}_{\mathcal{R}_1}[\mathcal{Z}(\mathcal{Q}_{\mathcal{R}_1}, \Upsilon)].$$

If we restrict the set of decision rules \mathcal{R}^* to the pair of rules $\{\mathcal{R}_d, \mathcal{R}_1\}$ then \mathcal{R}_1 can always be taken as the optimal decision rule. Thus, under Requirement 2.3.1.4, Requirement 2.3.1.3 is not fulfilled with such an "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$.

◇

Requirement 2.3.1.4 guarantees suitability of the "expectation" to a wide range of decision rules. The "full objectivity" of ordering of decision rules is approached if the "expectation" $\tilde{\mathcal{E}}_{\mathcal{R}}$ is made weakly dependent on the loss function considered. We have to adopt rather technical conditions on the set of such loss functions. Essentially, applicability to a very smooth functions and a restricted version of "linearity" of $\tilde{\mathcal{E}}_{\mathcal{R}}$ are required.

Requirement 2.3.1.6 (Independence of loss function) *Let us consider various loss functions $\mathcal{Z} \in \mathcal{Z}^*$. The "expectation" $\tilde{\mathcal{E}}$ acts on union of the sets $\mathcal{Z}_{\mathcal{R}^*}$ (2.1) of functions with a common uncertainty set Υ^* . The union is taken over the set of the loss functions*

$$\mathcal{Z}_{\mathcal{R}^*}^* \equiv \cup_{\mathcal{Z} \in \mathcal{Z}^*} \mathcal{Z}_{\mathcal{R}^*} \quad (2.4)$$

The set $\mathcal{Z}_{\mathcal{R}^*}^*$ is required to contain a subset of test loss functions that are zero out of a compact non-empty subset Ω of \mathcal{Q}^* and continuous on Ω .

The "expectation" is assumed to be an isotonic, sequentially continuous and boundedly uniformly continuous functional on $\mathcal{Z}_{\mathcal{R}^*}^*$ that is, moreover, additive on loss functions with non-overlapping supports

$$\tilde{\mathcal{E}}[\mathcal{Z}_1 + \mathcal{Z}_2] = \tilde{\mathcal{E}}[\mathcal{Z}_1] + \tilde{\mathcal{E}}[\mathcal{Z}_2] \text{ if } \mathcal{Z}_1 \mathcal{Z}_2 = 0, \mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{Z}_{\mathcal{R}^*}^*.$$

This technical Requirement allows us to get an integral representation of the "expectation" searched for. Its proof as well as definitions of the adopted non-common terms can be found in [13].

Proposition 2.3.1.7 (Integral form of "expectation") *Under Requirement 2.3.1.6, the "expectation" $\tilde{\mathcal{E}}$ has the form*

$$\tilde{\mathcal{E}}[\mathcal{Z}] = \int_{\Omega} \mathcal{U}(\mathcal{Z}(\mathcal{Q}), \mathcal{Q}) \mu(d\mathcal{Q}), \text{ where} \quad (2.5)$$

μ is a finite regular non-negative Borel measure on Ω . The utility function \mathcal{U} satisfies $\mathcal{U}(0, \mathcal{Q}) = 0$. It is continuous in values of $\mathcal{Z}(\cdot)$ almost everywhere (a.e.) on Ω , bounded a.e. on Ω for each \mathcal{Z} in the set of the test loss functions.

Remark(s) 2.3.1.8

1. The test loss functions are widely applicable and their consideration implies no practical restriction. The continuity requirements on $\tilde{\mathcal{E}}$ are also widely acceptable.

2. The linearity of $\tilde{\mathcal{E}}$ on functions with non-overlapping support seems to be sound. Any loss function $\mathcal{Z} \in \mathcal{Z}_{\mathcal{R}^*}^*$ can be written as $\mathcal{Z} = \mathcal{Z}\chi_\omega + \mathcal{Z}(1 - \chi_\omega) \equiv \mathcal{Z}_1 + \mathcal{Z}_2$, $\mathcal{Z}_1\mathcal{Z}_2 = 0$ with χ_ω denoting an indicator of a set $\omega \subset \Omega \subset \mathcal{Q}^*$. The indicator χ equals 1 on ω and it is zero outside of it.

The loss "expected" on the set ω and its complement should sum to the loss "expected" on the whole set of arguments.

3. The utility function \mathcal{U} allows the designer to express his/her attitude to the design consequences: he/she might be risk aware, risk prone or risk indifferent [12].
4. The utility function \mathcal{U} and the non-negative measure μ are universal for the whole set of test functions. \mathcal{U} and μ are (almost) "objective", i.e. suitable for a wide range of decision tasks facing the same uncertainty.

We formulate now our final objectivity oriented Requirement.

Requirement 2.3.1.9 (Indifference of the designer)

- The designer is risk indifferent, i.e. \mathcal{U} is the identity mapping.
- The "expectation" preserves any constant loss $\tilde{\mathcal{E}}[\text{constant}] = \text{constant}$.
- The involved measure μ has Radon-Nikodým derivative, [13], $f(\mathcal{Q})$ with respect to a dominating measure denoted $d\mathcal{Q}$.

In the treated cases, $d(\cdot)$ is either Lebesgue or counting measure.

Adopting Requirement 2.3.1.9, we get the basic representation Proposition that introduces objective "expectation".

Proposition 2.3.1.10 (Objective "expectation") Under Requirement 2.3.1.9, the "expectation" $\tilde{\mathcal{E}}$ (2.5) is formally identical with a mathematical expectation. The Radon Nikodým's derivative f , [13], has all properties of the joint probability (density) function (p(d)f) on \mathcal{Q}^* .

Proof: It is sufficient to observe that the preservation of constants implies that μ is a probabilistic measure, i.e. $\mu \geq 0$, $\mu(\mathcal{Q}^*) = 1$. ◇

Remark(s) 2.3.1.11

1. The (mathematical) "expectation" is distinguished by dropping the sign \sim as well as the quotation symbols " "

$$\mathcal{E}[\mathcal{Z}] \equiv \int_{\mathcal{Q}^*} \mathcal{Z}(\mathcal{Q})f(\mathcal{Q}) d\mathcal{Q} \equiv \int \mathcal{Z}(\mathcal{Q})f(\mathcal{Q}) d\mathcal{Q}. \quad (2.6)$$

2. The first item in Requirement 2.3.1.9 is our last step we made in the objectivity direction. It has clear meaning: we support objective, emotionally indifferent designers.

3. The last item in Requirement 2.3.1.9 is unnecessary but it covers all cases we are here interested in. It helps us to deal with simpler objects, namely, with probability density functions (pdf) or probability functions (pf).

Mostly, we make no formal distinction between these cases and we use notations related to pdfs. Only when necessary, we underline that we deal with a pf and write integrals as sums.

4. The notion of conditional expectation $\mathcal{E}[\bullet|*]$, [13] is adopted further on and it is treated in a simplistic way as the integral $\int \bullet(a)f(a|*) da$ weighted by the conditional pdf $f(a|*)$.

2.3.2 Calculus with pdfs

The joint pdf f on $\mathcal{Q} \equiv (\alpha, \beta, \gamma)$ is analysed/synthesised using several pdfs related to it. Let us recall meaning of pdfs derived from $f(\mathcal{Q})$.

Agreement 2.3.2.1 (Nomenclature of pdfs; Independence) We work with

Name	Meaning
joint pdf $f(\alpha, \beta \gamma)$ of α, β conditioned on γ	a pdf on $(\alpha, \beta)^*$ restricting $f(\mathcal{Q})$ on the cross-section of \mathcal{Q}^* given by a fixed γ
marginal pdf $f(\alpha \gamma)$ of α conditioned on γ	a pdf on α^* restricting $f(\mathcal{Q})$ on the cross-section of \mathcal{Q}^* given by a fixed γ with no information on β
marginal pdf $f(\beta \alpha, \gamma)$ of β conditioned on α, γ	a pdf on β^* restricting $f(\mathcal{Q})$ on the cross-section of \mathcal{Q}^* given by a fixed α, γ

The conditioning symbol $|$ is dropped if just trivial conditions are considered.

The pdf $f(\alpha, \beta)$ is the lower dimensional joint pdf (with respect to the pdf $f(\alpha, \beta, \gamma)$), $f(\beta)$ is the marginal pdf.

Quantities α and β are conditionally independent under the condition γ iff

$$f(\alpha, \beta|\gamma) = f(\alpha|\gamma)f(\beta|\gamma). \quad (2.7)$$

Our manipulations with the introduced pdfs rely on the following calculus.

Proposition 2.3.2.2 (Calculus with pdfs) For generic $(\alpha, \beta, \gamma) \in (\alpha, \beta, \gamma)^*$ it holds:

Non-negativity $f(\alpha, \beta|\gamma), f(\alpha|\beta, \gamma), f(\beta|\alpha, \gamma), f(\beta|\gamma) \geq 0$.

Normalization $\int f(\alpha, \beta|\gamma) d\alpha d\beta = \int f(\alpha|\beta, \gamma) d\alpha = \int f(\beta|\alpha, \gamma) d\beta = 1$.

Chain rule $f(\alpha, \beta|\gamma) = f(\alpha|\beta, \gamma)f(\beta|\gamma) = f(\beta|\alpha, \gamma)f(\alpha|\gamma)$.

Marginalization $f(\beta|\gamma) = \int f(\alpha, \beta|\gamma) d\alpha, f(\alpha|\gamma) = \int f(\alpha, \beta|\gamma) d\beta$.

Bayes rule

$$f(\beta|\alpha, \gamma) = \frac{f(\alpha|\beta, \gamma)f(\beta|\gamma)}{f(\alpha|\gamma)} = \frac{f(\alpha|\beta, \gamma)f(\beta|\gamma)}{\int f(\alpha|\beta, \gamma)f(\beta|\gamma) d\beta} \propto f(\alpha|\beta, \gamma)f(\beta|\gamma). \quad (2.8)$$

The proportion sign, \propto , means that the factor independent of β and uniquely determined by the normalization is not explicitly written in the equality represented.

Independence equivalents

$$f(\alpha, \beta|\gamma) = f(\alpha|\gamma)f(\beta|\gamma) \Leftrightarrow f(\alpha|\beta, \gamma) = f(\alpha|\gamma) \text{ or } f(\beta|\alpha, \gamma) = f(\beta|\gamma). \quad (2.9)$$

Proof: For motivation see [10], a more precise and more technical treatment exploits the measure theory [13]. Technically, an intermediate insight can be gained by considering loss functions dependent only on a part of \mathcal{Q} or with some parts of \mathcal{Q} "fixed by the condition", [9]. \diamond

Remark(s) 2.3.2.3

1. Duplicates in the presented formulas stress symmetry of many of them.
2. The technical fact that the identities, like (2.9), hold only almost everywhere, (a.e.) is mostly omitted in the subsequent explanations.
3. The Bayes rule (2.8) is a simple consequence of previous formulas. Its importance in this text justifies the explicit presentation in various forms.
4. The symmetric identity (2.9) says that β does not influence the description of α if α and β are conditionally independent for a given γ .

Often, a pdf $f(\alpha)$ of a multi-variate variable $\alpha \in \alpha^*$ is given and mapped on a variable β by a mapping $T : \alpha^* \rightarrow \beta^* \equiv T(\alpha^*)$. We need to specify the corresponding pdf $f(\beta)$. Rules for determining the pdf $f(\beta)$ are implied by the obvious need to preserve the expectation.

Proposition 2.3.2.4 (Pdfs of transformed quantities) *Let the expectation \mathcal{E} , acting on functions $\mathcal{B} : \beta^* \rightarrow (-\infty, \infty)$, be specified by the pdf $f_T(\beta)$, i.e.*

$$\mathcal{E}_T[\mathcal{B}] = \int \mathcal{B}(\beta) f_T(\beta) d\beta.$$

This functional expresses the same expectation as $\mathcal{E}[\mathcal{B}] = \int \mathcal{B}(T(\alpha)) f(\alpha) d\alpha$ iff

$$\int_{T(A)} f_T(T(\alpha)) dT(\alpha) = \int_A f(\alpha) d\alpha \quad (2.10)$$

for all measurable sets $A \subset \alpha^$.*

*Let α be a real vector, $\alpha \equiv [\alpha_1, \dots, \alpha_{\hat{\alpha}}]$ and $T = [T_1, \dots, T_{\hat{\alpha}}]$ bijection (one-to-one mapping) with finite continuous partial derivatives a.e. on α^**

$$J_{ij}(\alpha) \equiv \frac{\partial T_i(\alpha)}{\partial \alpha_j}, \quad i, j = 1, \dots, \hat{\alpha}, \quad (2.11)$$

for all entries T_i of T and entries α_j of α . Then,

$$f_T(T(\alpha)) |J(\alpha)| = f(\alpha), \quad \text{where} \quad (2.12)$$

$|\cdot|$ denotes determinant of the matrix in its argument.

Proof: Proposition describes substitutions in multi-variate integrals, see [13, 14]. \diamond

It is useful to summarize basic properties of expectation, which help us to simplify formal manipulations.

Proposition 2.3.2.5 (Basic properties of \mathcal{E}) For arbitrary functions $\mathcal{Z}_1(\cdot)$, $\mathcal{Z}_2(\cdot)$ on which the conditional expectation $\mathcal{E}[\cdot|\gamma]$ is well defined, $\mathcal{E}[\cdot|\gamma]$ has the following properties.

Isotonic nature of $\mathcal{E}[\cdot|\gamma]$: $\mathcal{Z}_1 \leq \mathcal{Z}_2$, cf. (2.3), $\Rightarrow \mathcal{E}[\mathcal{Z}_1|\gamma] \leq \mathcal{E}[\mathcal{Z}_2|\gamma]$.

Linearity of $\mathcal{E}[\cdot|\gamma]$: $\mathcal{E}[A(\gamma)\mathcal{Z}_1 + B(\gamma)\mathcal{Z}_2|\gamma] = A(\gamma)\mathcal{E}[\mathcal{Z}_1|\gamma] + B(\gamma)\mathcal{E}[\mathcal{Z}_2|\gamma]$
for arbitrary coefficients A, B depending at most on the condition γ .

Chain rule for expectation: $\mathcal{E}[\mathcal{E}[\cdot|\gamma, \zeta]|\gamma] = \mathcal{E}[\cdot|\gamma]$ for an arbitrary additional condition ζ .

Conditional covariance of a vector α $\text{cov}[\alpha|\gamma] \equiv \mathcal{E}[(\alpha - \mathcal{E}[\alpha|\gamma])(\alpha - \mathcal{E}[\alpha|\gamma])'|\gamma]$ is related to the non-central moments through the formula

$$\text{cov}[\alpha|\gamma] = \mathcal{E}[\alpha\alpha'|\gamma] - \mathcal{E}[\alpha|\gamma]\mathcal{E}[\alpha'|\gamma], \quad ' \text{ is transposition.} \quad (2.13)$$

Jensen inequality bounds expectation of a convex function $T_\gamma : \alpha^* \rightarrow (-\infty, \infty)$

$$\mathcal{E}[T_\gamma(\alpha)|\gamma] \geq T_\gamma(\mathcal{E}[\alpha|\gamma]). \quad (2.14)$$

Proof: All statements can be verified by using the integral expression (2.6) of the expectation. Proof of Jensen inequality can be found e.g. in [15]. \diamond

Remark(s) 2.3.2.6

1. Proposition is formulated for the conditional expectation. The unconditional case is formally obtained by omitting the condition used.
2. Note that whenever the expectation is applied to a vector (matrix) function V it should be understood as the vector (matrix) of expectations

$$[\mathcal{E}(V)]_i \equiv \mathcal{E}(V_i). \quad (2.15)$$

2.3.3 Basic decision-making lemma

The optimal selection of *admissible decision rules* relies on the following key proposition that converts minimization over mappings to an “ordinary” minimization.

Proposition 2.3.3.1 (Basic decision-making lemma) The optimal admissible decision rule \mathcal{R}^o

$$\mathcal{R}^o(\mathcal{P}_a) \equiv a^o(\mathcal{P}_a), \quad \forall \mathcal{P}_a \in \mathcal{P}_a^*$$

minimizing the expected loss (2.6) can be constructed value-wise as follows.

To each $\mathcal{P}_a \in \mathcal{P}_a^*$, a minimizing argument $a^o(\mathcal{P}_a)$ in

$$\min_{a \in a^*} \mathcal{E}[\mathcal{Z}(\mathcal{P}_a, a, \mathcal{F}_a)|a, \mathcal{P}_a] \quad (2.16)$$

is assigned as the value of the optimal decision rule corresponding to the considered argument. The reached minimum is

$$\min_{\{\mathcal{R}: \mathcal{P}_a^* \rightarrow a^*\}} \mathcal{E}[\mathcal{Z}(\mathcal{P}_a, a, \mathcal{F}_a)] = \mathcal{E} \left\{ \min_{a \in a^*} \mathcal{E}[\mathcal{Z}(\mathcal{P}_a, a, \mathcal{F}_a)|a, \mathcal{P}_a] \right\}. \quad (2.17)$$

Proof: Let us fix an arbitrary $\mathcal{P}_a \in \mathcal{P}_a^*$. The definition of minimum implies

$$\mathcal{E}[\mathcal{Z}(\mathcal{P}_a, \mathcal{R}^o(\mathcal{P}_a), \mathcal{F}_a) | \mathcal{R}^o(\mathcal{P}_a), \mathcal{P}_a] \leq \mathcal{E}[\mathcal{Z}(\mathcal{P}_a, a, \mathcal{F}_a) | a, \mathcal{P}_a], \quad \forall a \in a^*.$$

Let an admissible rule $\mathcal{R} : \mathcal{P}_a^* \rightarrow a^*$ assign a decision $a \in a^*$ to the considered \mathcal{P}_a . Then, the previous inequality can be written

$$\mathcal{E}[\mathcal{Z}(\mathcal{P}_a, \mathcal{R}^o(\mathcal{P}_a), \mathcal{F}_a) | \mathcal{R}^o(\mathcal{P}_a), \mathcal{P}_a] \leq \mathcal{E}[\mathcal{Z}(\mathcal{P}_a, \mathcal{R}(\mathcal{P}_a), \mathcal{F}_a) | \mathcal{R}(\mathcal{P}_a), \mathcal{P}_a].$$

We apply unconditional expectation $\mathcal{E}[\cdot]$ acting on functions of \mathcal{P}_a to this inequality. Due to the isotonic nature of \mathcal{E} , the inequality is preserved. The chain rule for expectations, see Proposition 2.3.2.5, implies that on the left hand side of the resulting inequality the value of the unconditional expected loss for \mathcal{R}^o is found. \mathcal{R}^o assigns to each $\mathcal{P}_a \in \mathcal{P}_a^*$ the decision $a^o(\mathcal{P}_a)$. On the right hand side of the discussed inequality, we get the unconditional expected loss corresponding to an arbitrarily chosen $\mathcal{R} : \mathcal{P}_a^* \rightarrow a^*$. \diamond

Proposition and its proof imply no preferences if there are more absolutely minimizing arguments $a^o(\mathcal{P}_a)$. We can use any of them or switch between them in a random manner whenever the pdf $f(a_t | \mathcal{P}_{a_t})$ has its support concentrated on them. This is an example where a *randomized causal strategy* may occur. We specify it formally as it is used later on more extensively.

Agreement 2.3.3.2 (Outer model of randomized decision strategy) *The pdf $f(a | \mathcal{P}_a)$ is called outer model of the decision rule. The collection of pdfs $\{f(a_t | \mathcal{P}_{a_t})\}_{t \in t^*}$ forms the outer model of the decision strategy.*

A decision rule $f(a | \mathcal{P}_a)$ is called randomized decision rule if its support contains at least two different values of a_t . The strategy is called randomized strategy if some of its rules is randomized.

Remark(s) 2.3.3.3

1. *We do not enter the technical game with ε -optimum: the existence of the various minimizing arguments is implicitly supposed.*
2. *It is worth repeating that the optimal decision rule is constructed value-wise. Formally, the minimization should be performed for all possible instances of experience $\mathcal{P}_a \in \mathcal{P}_a^*$ in order to get the decision rule.*

Often, we are interested in the optimal decision for a given fixed, say observed, experience. Then, just a single minimization is necessary. This is typically the case of the estimation problem. This possibility makes the main distinction from the dynamic design, when optimal strategy, a sequence of decision rules, is searched for. In this case, discussed in next Section, the construction of decision rules is necessary. This makes the dynamic design substantially harder and mostly infeasible [5, 6].

2.4 Dynamic design

We are searching for the optimal admissible strategy. We consider the usual case, when its individual rules use a non-decreasing sequence of experience. The extending experience models increasing amount of data available for the decision-making.

2.4.1 Dynamic programming

The optimal admissible strategy can be found by using a stochastic version of celebrated *dynamic programming* [16]. It is nothing but a repetitive application of Proposition 2.3.3.1.

Proposition 2.4.1.1 (Stochastic dynamic programming) *The optimal causal strategy $\{\mathcal{R}_t^o : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*} \in \{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}$ acting on non-decreasing sequence of experience $\mathcal{P}_{a_t} \subset \mathcal{P}_{a_{t+1}}$ and minimizing the expected loss function $\mathcal{E}[\mathcal{Z}(\mathcal{Q})]$ can be constructed in a value-wise way. For every $t \in t^*$ and each $\mathcal{P}_{a_t} \in \mathcal{P}_{a_t}^*$, it is sufficient to take a minimizing argument $a^o(\mathcal{P}_{a_t})$ in*

$$\mathcal{V}(\mathcal{P}_{a_t}) = \min_{a_t \in a_t^*} \mathcal{E}[\mathcal{V}(\mathcal{P}_{a_{t+1}})|a_t, \mathcal{P}_{a_t}], \quad t \in t^* \quad (2.18)$$

as the decision generated by the t -th rule of the optimal strategy, i.e. $a^o(\mathcal{P}_{a_t}) = \mathcal{R}_t^o(\mathcal{P}_{a_t})$.

The recursion (2.18) is performed in the backward manner against the course given by the increasing experience. It starts with

$$\mathcal{V}(\mathcal{P}_{a_{i+1}}) \equiv \mathcal{Z}(\mathcal{Q}). \quad (2.19)$$

The reached minimum has the value $\mathcal{E}[\mathcal{V}(\mathcal{P}_{a_1})] = \min_{\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}} \mathcal{E}[\mathcal{Z}(\mathcal{Q})]$.

Proof: Let us define $\mathcal{P}_{a_{i+1}} \equiv \mathcal{Q}$ in order to get a uniform notation It is legitimate as a_{i+1} is not optimized. The definition of minimum and Proposition 2.3.3.1 imply

$$\begin{aligned} \min_{\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}} \mathcal{E}[\mathcal{Z}(\mathcal{P}_{a_{i+1}})] &= \min_{\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t < i}} \left\{ \min_{\mathcal{R}_i : \mathcal{P}_{a_i}^* \rightarrow a_i^*} \mathcal{E}[\mathcal{Z}(\mathcal{P}_{a_{i+1}})] \right\} \stackrel{(2.17)}{=} \\ &= \min_{\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t < i}} \mathcal{E} \left[\min_{a_i \in a_i^*} \mathcal{E}[\mathcal{Z}(\mathcal{P}_{a_{i+1}})|a_i, \mathcal{P}_{a_i}] \right]. \end{aligned}$$

Denoting $\mathcal{V}(\mathcal{P}_{a_i}) \equiv \min_{a_i \in a_i^*} \mathcal{E}[\mathcal{Z}(\mathcal{P}_{a_{i+1}})|a_i, \mathcal{P}_{a_i}]$, we proved the first step of the recursion and specified the start (2.19). The following step becomes

$$\min_{\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t < i}} \mathcal{E} \left[\mathcal{V}(\mathcal{P}_{a_i}) \right].$$

We face the identical situation as above with the horizon decreased by one. Thus, the procedure can be repeated till the initial rule \mathcal{R}_1 is constructed. \diamond

The optimization relies on our ability to evaluate the expectations

$$\mathcal{E}[\mathcal{V}(\mathcal{P}_{a_{t+1}})|a_t, \mathcal{P}_{a_t}] = \int_{\Delta_{a_t}^*} \mathcal{V}(\mathcal{P}_{a_t}, a_t, \Delta_{a_t}) f(\Delta_{a_t} | a_t, \mathcal{P}_{a_t}) d\Delta_{a_t}, \quad \forall t \in t^*.$$

The introduced *innovation* Δ_{a_t} contains those observable quantities that cannot be used for the choice of a_t . They are not in \mathcal{P}_{a_t} but they belong to $\mathcal{P}_{a_{t+1}} \setminus a_t$. The pdfs $\{f(\Delta_{a_t} | a_t, \mathcal{P}_{a_t})\}_{t \in t^*}$ model the relationships of Δ_{a_t} to \mathcal{P}_t and a_t .

Agreement 2.4.1.2 (Outer model of the system) *The collection of pdfs*

$$\{f(\Delta_{a_t} | a_t, \mathcal{P}_{a_t})\}_{t \in t^*}, \quad (2.20)$$

required for the optimal design, is called outer model of the system.

Remark(s) 2.4.1.3

1. The term *outer model of the system* is abbreviated to the model of the system or even to the model. The exact meaning is clear from the specific context.
2. Often, the innovation $\Delta_{a_t} = y_t =$ observable output of the system. The set-point to which the output should be driven by the chosen input u_t has to be included into Δ_{u_t} if its values are uncertain.

The following agreement is used in a presentation of the most common version of dynamic programming.

Agreement 2.4.1.4 (Data driven design; internal quantities) *The design is called data driven if realization \mathcal{Q} consists of optional and potentially observable quantities, i.e. with ignorance consisting of unobserved data only*

$$\mathcal{Q} \equiv (\Delta(\dot{t}), a(\dot{t})) \equiv (\mathcal{P}_a, a, \mathcal{F}_a). \quad (2.21)$$

Quantities in \mathcal{Q} that are never observed directly are called internal quantities.

Proposition 2.4.1.5 (Dynamic programming for additive loss function) *Let us consider data driven design and search for the optimal admissible strategy $\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}$ acting on a non-decreasing sequence of experience $\{\mathcal{P}_{a_t}\}_{t \in t^*}$.*

Then, the optimal strategy $\{\mathcal{R}_t^o : \mathcal{P}_{a_t}^ \rightarrow a_t^*\}_{t \in t^*}$ minimizing the expected additive loss function*

$$\mathcal{E} \left[\sum_{t \in t^*} z(\Delta(t), a(t)) \right] \quad (2.22)$$

can be constructed value-wise. For all $t \in t^$ and to each $\mathcal{P}_{a_t} \in \mathcal{P}_{a_t}^*$, a minimizing argument $a^o(\mathcal{P}_{a_t}) = \mathcal{R}_t^o(\mathcal{P}_{a_t})$ in*

$$\mathcal{V}(\mathcal{P}_{a_t}) = \min_{a_t \in a_t^*} \mathcal{E}[z(\Delta(t), a(t)) + \mathcal{V}(\mathcal{P}_{a_{t+1}})|a_t, \mathcal{P}_{a_t}], \quad t \in t^* \quad (2.23)$$

is assigned. The recursion (2.23) is performed in the backward manner against the course given by the increasing experience, starting from

$$\mathcal{V}(\mathcal{P}_{a_{i+1}}) \equiv 0. \quad (2.24)$$

The reached minimum has the value

$$\min_{\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}} \mathcal{E}[\mathcal{Z}(\mathcal{P}_{a_{i+1}})] = \mathcal{E}[\mathcal{V}(\mathcal{P}_{a_1})].$$

Proof: It follows exactly the line of Proposition 2.4.1.1 with a modified definition of the function $\mathcal{V}(\cdot)$

$$\mathcal{V}(\mathcal{P}_{a_t}) \equiv \min_{\{\mathcal{R}_\tau : \mathcal{P}_{a_\tau}^* \rightarrow a_\tau^*\}_{\tau \geq t}} \mathcal{E} \left[\sum_{\tau \geq t} z(\Delta(\tau), a(\tau)) | a_t = \mathcal{R}_t(\mathcal{P}_{a_t}), \mathcal{P}_{a_t} \right]. \quad (2.25)$$

◇

Agreement 2.4.1.6 (Partial loss; Bellman function) *The function $z(\Delta(t), a(t))$ is called the partial loss. The function $\mathcal{V}(\cdot)$ occurring in dynamic programming is called Bellman function. Bellman function $\mathcal{V}(\cdot)$ in (2.25) is also called the optimal loss-to-go.*

2.4.2 Fully probabilistic design

A specific but rather general design is formulated and solved here. It is believed to form a bridge between optimal and practically optimal design.

Here, and in many other places, we need the notion of the *Kullback-Leibler distance* [17] that measures well proximity of a pair of pdfs.

Agreement 2.4.2.1 (Kullback-Leibler distance) *Let f, g be a pair of pdfs acting on a common set x^* . Then, the Kullback-Leibler distance $\mathcal{D}(f||g)$ is defined by the formula*

$$\mathcal{D}(f||g) \equiv \int_{x^*} f(x) \ln \left(\frac{f(x)}{g(x)} \right) dx. \quad (2.26)$$

For conciseness, the Kullback-Leibler distance is referred to as the KL distance.

Proposition 2.4.2.2 (Basic properties of KL distance)

Let f, g be a pair of pdfs acting on a same set. It holds

1. $\mathcal{D}(f||g) \geq 0$,
2. $\mathcal{D}(f||g) = 0$ iff $f = g$ (a.e.),
3. $\mathcal{D}(f||g) = \infty$ iff on a set of a positive measure $f > 0$ and $g = 0$,
4. $\mathcal{D}(f||g) \neq \mathcal{D}(g||f)$; the KL distance does not obey triangle inequality.

Proof: See, for instance, [15] ◇

Now we are ready to formulate and solve the fully probabilistic design problem. A simple version is presented here considering the data-driven design, see Agreement 2.4.1.4. In this case, the joint pdf $f(\mathcal{Q}) \equiv f(\Delta(\dot{t}), a(\dot{t}))$ describing observable quantities of interest can be factorized by a repetitive use of the chain rule, see Proposition 2.3.2.2,

$$f(\Delta(\dot{t}), a(\dot{t})) = \prod_{t \in t^*} f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}) f(a_t|\mathcal{P}_{a_t}). \quad (2.27)$$

The first factors $\{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})\}_{t \in t^*}$ under the product sign describe possible reactions of the system on the decision a_t under the experience \mathcal{P}_{a_t} . These pdfs form the outer model of the system, see Agreement 2.4.1.2. Similarly, $\{f(a_t|\mathcal{P}_{a_t})\}_{t \in t^*}$ represent an outer model of a randomized decision strategy to be chosen, see Agreement 2.3.3.2. Looking at the joint pdf (2.27), it seems to be “natural” to formulate the design as an attempt to make this pdf as close as possible to some “ideal” joint pdf.

Agreement 2.4.2.3 (Fully probabilistic design) *The fully probabilistic, data driven, design specifies its target through an ideal pdf*

$${}^{[I]}f(\mathcal{Q}) = \prod_{t \in t^*} {}^{[I]}f(\Delta_{a_t}|a_t, d(t-1)) {}^{[I]}f(a_t|d(t-1)).$$

The optimal admissible, possibly randomized, decision strategy is defined as a minimizer of the KL distance (2.26) of $f(d(\dot{t})) = f(\Delta(\dot{t}), a(\dot{t}))$ and ${}^{[I]}f(d(\dot{t})) = {}^{[I]}f(\Delta(\dot{t}), a(\dot{t}))$

$$\mathcal{D}(f||{}^{[I]}f) \equiv \int f(\Delta(\dot{t}), a(\dot{t})) \ln \left(\frac{f(\Delta(\dot{t}), a(\dot{t}))}{{}^{[I]}f(\Delta(\dot{t}), a(\dot{t}))} \right) d(\Delta(\dot{t}), a(\dot{t})). \quad (2.28)$$

Proposition 2.4.2.4 (Solution of fully probabilistic design) *The optimal strategy minimizing KL distance (2.28) has the form*

$$\begin{aligned}
f(a_t|d(t-1)) &= {}^{[I]}f(a_t|d(t-1)) \frac{\exp[-\omega_\gamma(a_t, d(t-1))]}{\gamma(d(t-1))}, \quad \text{where} & (2.29) \\
\gamma(d(t-1)) &\equiv \int_{a_t^*} {}^{[I]}f(a_t|d(t-1)) \exp[-\omega(a_t, d(t-1))] da_t \\
\omega_\gamma(a_t, d(t-1)) &\equiv \int_{\Delta_t^*} f(\Delta_t|a_t, d(t-1)) \ln \left(\frac{f(\Delta_t|a_t, d(t-1))}{\gamma(d(t)) {}^{[I]}f(\Delta_t|a_t, d(t-1))} \right) d\Delta_t \\
\gamma(d(\hat{t})) &= 1. & (2.30)
\end{aligned}$$

The solution is performed against the time course, starting at $t = \hat{t}$.

Proof: Using chain rule, we can write the KL distance in the form

$$\begin{aligned}
\mathcal{D}(f||{}^{[I]}f) &= \mathcal{E} \left\{ \sum_{t \in \hat{t}^*} \int f(a_t|d(t-1)) \left[\ln \left(\frac{f(a_t|d(t-1))}{{}^{[I]}f(a_t|d(t-1))} \right) + \omega(a_t, d(t-1)) \right] da_t \right\} \text{ with} \\
\omega(a_t, d(t-1)) &\equiv \int f(\Delta_t|a_t, d(t-1)) \ln \left(\frac{f(\Delta_t|a_t, d(t-1))}{{}^{[I]}f(\Delta_t|a_t, d(t-1))} \right) d\Delta_t. \text{ Let us denote } \ln(\gamma(d(t))) \equiv \\
&\equiv \min_{\{f(a_{\tau+1}|d(\tau))\}_{\tau=t}^{\hat{t}}} \mathcal{E} \left\{ \sum_{\tau=t+1}^{\hat{t}} \int f(a_\tau|d(\tau-1)) \left[\ln \left(\frac{f(a_\tau|d(\tau-1))}{{}^{[I]}f(a_\tau|d(\tau-1))} \right) + \omega(a_\tau, d(\tau-1)) \right] da_\tau|d(t) \right\}.
\end{aligned}$$

Then, this definition implies that $\gamma(d(\hat{t})) = 1$ and

$$\begin{aligned}
-\ln(\gamma(d(t))) &\equiv \min_{f(a_{t+1}|d(t))} \int f(a_{t+1}|d(t)) \left[\ln \left(\frac{f(a_{t+1}|d(t))}{{}^{[I]}f(a_{t+1}|d(t))} \right) + \omega_\gamma(a_{t+1}, d(t)) \right] da_{t+1} \text{ with} \\
\omega_\gamma(a_{t+1}, d(t)) &\equiv \int f(\Delta_t|a_t, d(t-1)) \ln \left(\frac{f(\Delta_t|a_t, d(t-1))}{\gamma(d(t)) {}^{[I]}f(\Delta_t|a_t, d(t-1))} \right) d\Delta_t. \text{ It implies} \\
-\ln(\gamma(d(t))) &\equiv \min_{f(a_{t+1}|d(t))} \int f(a_{t+1}|d(t)) \\
&\left[\ln \left(\frac{f(a_{t+1}|d(t))}{{}^{[I]}f(a_{t+1}|d(t)) \exp[-\omega_\gamma(a_{t+1}, d(t))]} \right) da_{t+1} - \ln \left(\int {}^{[I]}f(a_{t+1}|d(t)) \exp[-\omega_\gamma(a_{t+1}, d(t))] da_{t+1} \right) \right].
\end{aligned}$$

The first term in the above identity is the KL distance, that reaches its smallest zero value for the claimed pdf. At the same time, it defines the form of the reached minima. An alternative derivations, with more details, see [18, 19]. \diamond

Remark(s) 2.4.2.5

1. At a descriptive level, the stochastic dynamic programming consists of a sequence of the evaluation pairs

(take conditional expectation, minimize).

Except of a few numerically solvable cases, some approximation techniques have to be employed. The complexity of the approximated optimum prevents a systematic use of the standard approximation theory and various ad hoc schemes are adopted.

The fully probabilistic design finds minimizers explicitly and reduces the design to a sequence of conceptually feasible multi-variate integrations.

2. The found optimal strategy is randomized and obviously causal one. The technological restriction are met trivially if the chosen ideal strategy meets them, if $\text{supp} \left[{}^{[t]}f(a_t | \mathcal{P}_{a_t}) \right] \subset a_t^*$, cf. (2.29).

2.4.3 Asymptotic of design

The analysis outlined here serves us mainly for interpretation purposes. Thus, all technicalities are suppressed as much as possible.

The asymptotic of the dynamic programming is analyzed for horizon $\dot{t} \rightarrow \infty$ within this section. The data-driven case with an additive loss function (2.22) is considered. Note that the general loss function can always be converted in it by defining

$$z(\Delta(t), a(t)) = \begin{cases} \mathcal{Z}(\Delta(\dot{t}), a(\dot{t})) & \text{if } t = \dot{t} \\ 0 & \text{otherwise} \end{cases}. \quad (2.31)$$

We deal, however, with a simpler but still useful case by assuming that

- there is a finite-dimensional information state, i.e. $\mathcal{P}_{a_t} \equiv x_{t-1} \equiv$ a finite-dimensional vector,
- the partial loss depends on x_t and a_t only $z(\Delta(t), a(t)) \equiv z(x_t, a_t)$, i.e. the considered loss is

$$\mathcal{Z}(\Delta(\dot{t}), a(\dot{t})) = \sum_{t \in \dot{t}^*} z(x_t, a_t).$$

Agreement 2.4.3.1 (Stabilizing strategy) *Let us consider sequence of decision-making problems with the growing horizon $\dot{t} \rightarrow \infty$. The strategy $\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in \dot{t}^* \equiv \{1, \dots, \dot{t}\}} \forall \dot{t} \in \{1, 2, \dots\}$ is called stabilizing strategy if there is a finite constant c such that*

$$\mathcal{E}[z(x_t, a_t) | a_t, \mathcal{P}_{a_t}] \leq c < \infty, \quad t \in \{1, 2, 3, \dots\}. \quad (2.32)$$

Proposition 2.4.3.2 (Asymptotic design) *Let a stabilizing strategy exist. Then there is an optimal, stationary strategy, formed by a repetitive use of the same rule, for $\dot{t} \rightarrow \infty$ whose decisions are minimizing arguments in the formal analogy of (2.23)*

$${}^{[\infty]}\mathcal{V}(x_{t-1}) + C = \min_{a_t \in a_t^*} \mathcal{E} \left[z(x_t, a_t) + {}^{[\infty]}\mathcal{V}(x_t) | a_t, x_{t-1} \right]. \quad (2.33)$$

with a constant $C \leq c$ and a time invariant Bellman function ${}^{[\infty]}\mathcal{V}(x)$.

Proof: CCC >> MK: this proof has to be modified << CCC (Outline) Let us take any finite horizon \dot{t} . We denote ${}^{[\dot{t}]}\mathcal{E}$ the expectation taken for the strategy $\{{}^{[\dot{t}]}\mathcal{R}_t\}_{t \in \dot{t}^*}$ optimal for the additive loss specified on this horizon. Similarly, ${}^{[\dot{t}]}\tilde{\mathcal{V}}(\mathcal{P}_{a_t}) \equiv {}^{[\dot{t}]}\tilde{\mathcal{V}}(x_{t-1})$ denotes the optimal loss-to-go, see Agreement 2.4.1.6 and ${}^{[\dot{t}]}a_t = {}^{[\dot{t}]}\mathcal{R}_t(x_{t-1})$.

The definition of optimum, the additive form of the loss (2.22) and the assumption (2.32) imply that

$$c \geq [t]C \equiv [t]\mathcal{E}[\mathcal{Z}(\Delta(\overset{\circ}{t}), [t]a(\overset{\circ}{t}))] - [t-1]\mathcal{E}[\mathcal{Z}(\Delta(\overset{\circ}{t}-1), [t-1]a(\overset{\circ}{t}-1))] \geq 0.$$

Consequently,

$$\limsup_{t \rightarrow \infty} \left\{ [t]\mathcal{E}[\mathcal{Z}(\Delta(\overset{\circ}{t}), [t]a(\overset{\circ}{t}))] - \sum_{t=1}^{\overset{\circ}{t}} [t]C \right\} \equiv [\infty]\mathcal{V} < \infty.$$

If we redefine the partial loss to $z(x_t, a_t) - [t]C$ then the optimal strategies for all $\overset{\circ}{t}$ do not change and $[t]\mathcal{E}[\mathcal{Z}(\Delta(\overset{\circ}{t}), [t]a(\overset{\circ}{t}))] = [\infty]\mathcal{V} + o(\overset{\circ}{t})$, where the symbols $o(\overset{\circ}{t})$ means, as usual, a term converging to zero for $\overset{\circ}{t} \rightarrow \infty$.

Similar considerations apply to the modified optimal loss-to-go $[t]\mathcal{V}(x_t)$, Agreement 2.4.1.6, imply that (with the modified partial loss!) it converges for all $x_t \in x_t^*$ to some $[\infty]\mathcal{V}(x_t)$.

With the original partial loss and modified loss-to-go, the dynamic programming (2.23) gets the form

$$[t]\mathcal{V}(x_{t-1}) + [t]C = \min_{a_t \in a_t^*} [t]\mathcal{E} \left[z(x_t, a_t) + [t]\mathcal{V}(x_t) | a_t, x_{t-1} \right] \text{ that converges for } \overset{\circ}{t} \rightarrow \infty.$$

◇

Remark(s) 2.4.3.3

1. The same proof is directly applicable to the fully probabilistic design as it can be seen as an instance of the additive loss function.
2. Solutions of the Bellman equation obtained for a growing finite horizon $\overset{\circ}{t}$ can be interpreted as successive approximations for solving its stationary counterpart (2.33).
3. So called iterations in strategy space [20] is an alternative and efficient way of finding the asymptotic solution. Essentially, a stabilizing stationary, i.e. repeating a single rule, strategy $\{\mathcal{R}\}$ is selected and the linear equation

$$\mathcal{V}(x) + C = \mathcal{E}[z(\tilde{x}, \mathcal{R}(x)) + \mathcal{V}(\tilde{x}) | \mathcal{R}(x), x]$$

is solved for the function $\mathcal{V}(\cdot)$ and constant C . Then, a new approximating strategy is found value-wise $\mathcal{R}(x) = \text{Arg} \min_{a \in a^*} \mathcal{E}[z(\tilde{x}, a) + \mathcal{V}(\tilde{x}) | a, x]$ with such a $\mathcal{V}(\cdot)$. Under general conditions, the newly found strategy is stabilizing and iterations may be repeated till the guaranteed convergence. Details of this procedure are out of scope of this work but it should be considered when searching for efficient numerical procedures.

2.5 Learning

Generally, considered behavior \mathcal{Q}^* contains quantities that are never observed directly and, in spite of this, we want to describe them or influence them. We called them internal quantities, see Agreement 2.4.1.4. Even in this case, the optimal decision-making needs the outer model (2.20), see Proposition 2.4.1.1, that should reflect them. Here we describe how to get it. The presented construction is of an independent interest as a good *formal model of learning*.

2.5.1 Bayesian filtration

This section deals with a general case when the construction of the outer model relies on filtering, i.e. learning of *time variant* internal, i.e. directly unobservable, quantities. It requires the following elements.

Requirement 2.5.1.1 (Models; natural conditions of decision-making)

1. The innovations Δ_{a_t} are related to experience \mathcal{P}_{a_t} and decisions a_t through the observation model

$$\{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta_t)\}_{t \in t^*} \quad (2.34)$$

that is given up to unknown internal quantities $\Theta_t \in \Theta_t^* \subset \mathcal{F}_{a_\tau}, \forall \tau \in t^*$.

2. The evolution of the quantities $\Theta(\dot{t}) \in \Theta^*(\dot{t})$ is described by a known collection of pdfs called the time evolution model

$$\{f(\Theta_t|a_t, \mathcal{P}_{a_t}, \Theta_{t-1})\}_{t \in t^*}. \quad (2.35)$$

3. The quantities $\Theta(\dot{t})$ are unknown to the strategies considered. The natural conditions of decision-making (a slight generalization of natural conditions of control [10]) express it formally. They postulate independence of a_t and Θ_t when conditioned on \mathcal{P}_{a_t}

$$f(a_t|\mathcal{P}_{a_t}, \Theta_t) = f(a_t|\mathcal{P}_{a_t}) \quad \Leftrightarrow \quad f(\Theta_t|a_t, \mathcal{P}_{a_t}) = f(\Theta_t|\mathcal{P}_{a_t}). \quad (2.36)$$

Proposition 2.3.2.2

4. The initial values of $\Theta_0, \mathcal{P}_{a_1}$ add nothing new to the prior information so that the prior pdf

$$f(\Theta_1) \equiv f(\Theta_1|a_1, \mathcal{P}_{a_1}, \Theta_0) = f(\Theta_1|a_1, \mathcal{P}_{a_1}) \underbrace{=}_{(2.36)} f(\Theta_1|\mathcal{P}_{a_1}) \quad (2.37)$$

is the first term in the sequence of pdfs (2.35).

Remark(s) 2.5.1.2

1. Often, the unknown quantities Θ_t together with the decision a_t are assumed to describe the involved conditional pdfs fully. Then, \mathcal{P}_{a_t} can be omitted and Θ_t can be identified with the information state.
2. The natural conditions of decision-making express the assumption that $\Theta_t \notin \mathcal{P}_{a_\tau} \forall \tau \in t^*$. Thus, values of Θ_t cannot be used by the decision rules forming admissible strategy. Alternatively, see (2.36), we cannot gain information about Θ_t from the decision a_t if the corresponding innovation Δ_{a_t} (the corresponding reaction of the system) is not available.

The natural conditions of decision-making are “naturally” fulfilled by strategies we are designing. They have to be checked when data influenced by an “externally chosen” strategy are processed.

Proposition 2.5.1.3 (Generalized Bayesian filtering) *Let Requirement 2.5.1.1 be met. Then, the outer model of the system (2.20) is given by the formula*

$$f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}) = \int_{\Theta_t^*} f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta_t) f(\Theta_t|\mathcal{P}_{a_t}) d\Theta_t. \quad (2.38)$$

The evolution of the pdf $f(\Theta_t|\mathcal{P}_{a_t})$, called (generalized Bayesian) filtration of unknown quantities Θ_t , is described by the following recursion that starts from the prior pdf $f(\Theta_1)$:

- Data updating

$$f(\Theta_t|\mathcal{P}_{a_{t+1}}) = \frac{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta_t) f(\Theta_t|\mathcal{P}_{a_t})}{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})} \propto f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta_t) f(\Theta_t|\mathcal{P}_{a_t}) \quad (2.39)$$

that incorporates the innovation Δ_{a_t} and the decision a_t , and

- Time updating

$$f(\Theta_{t+1}|\mathcal{P}_{a_{t+1}}) = \int_{\Theta_t^*} f(\Theta_{t+1}|a_{t+1}, \mathcal{P}_{a_{t+1}}, \Theta_t) f(\Theta_t|\mathcal{P}_{a_{t+1}}) d\Theta_t \quad (2.40)$$

that reflects the time evolution $\Theta_t \rightarrow \Theta_{t+1}$.

Proof: Sequential use of marginalization, chain rule, Proposition 2.3.2.2, and of the natural conditions of decision making (2.36) implies (2.38)

$$\begin{aligned} f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}) &= \int_{\Theta_t^*} f(\Delta_{a_t}, \Theta_t|a_t, \mathcal{P}_{a_t}) d\Theta_t = \int_{\Theta_t^*} f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta_t) f(\Theta_t|a_t, \mathcal{P}_{a_t}) d\Theta_t = \\ &= \int_{\Theta_t^*} f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta_t) f(\Theta_t|\mathcal{P}_{a_t}) d\Theta_t. \end{aligned}$$

Data updating coincides with the Bayes rule in which the outer model of the strategy cancels as it does not depend on Θ_t due to the natural conditions of decision making (2.36).

Marginalization, chain rule and natural conditions of decision making imply also the formula for time updating. \diamond

Agreement 2.5.1.4 (Filtering; predictive pdf) *The process of generating filtration is called (generalized Bayesian) filtering. The outer model of the system obtained by filtering is called predictive pdf.*

Remark(s) 2.5.1.5

1. The term predictive pdf reflects the way how the outer model of the system has been obtained. It uses the observed experience and extrapolates it into ignorance assuming that the mechanism of generating Θ_t does not change.

This accumulation of experience and its extrapolation represent a good formal model of learning.

2. It has to be stressed that the accumulation of experience can take place only when the rules governing the behavior are not changed during it, i.e. when we can rely on the validity of the underlying models.

3. The construction of the predictive pdf is our key motivation for filtering. Its results are often of an independent interest.
4. Under the natural conditions of decision-making, the filtering relies on the knowledge of decisions and not on the knowledge of rules $\mathcal{R} : \mathcal{P}_a^* \rightarrow a^*$ generating them. It is practically important when we learn while decision loop is closed, especially, when decisions are made by a human decision maker.
5. The time evolution model $f(\Theta_t|\Theta_{t-1}, a_t, \mathcal{P}_{a_t})$ as well as the observation model $f(\Delta_{a_t}|\Theta_t, a_t, \mathcal{P}_{a_t})$ have to result from a theoretical modelling of the system in question. Such a modelling uses both field knowledge, like laws of conservation, and approximation capabilities of the selected family of the models involved. Often, deterministic relationships are modelled and then the “deviations” from an “expected” trajectory are described.
6. The prior pdf $f(\Theta_1)$ allows us to introduce an information based on expert knowledge or analogy to situations observed previously.
7. The observed data, the only bridge to reality, enter the evaluations in the data updating step only when the newest innovation-decision pair is processed. This simple observation is important for approximation of the time evolution model, see Section 4.2.2.
8. In summary, the described Bayesian filtering combines prior information in $f(\Theta_1)$, theoretical knowledge of the specific fields described by $f(\Delta_{a_t}|\Theta_t, a_t, \mathcal{P}_{a_t})$, $f(\Theta_t|\Theta_{t-1}, a_t, \mathcal{P}_{a_t})$ and observed data $d(\overset{\circ}{t}) = (\Delta(\overset{\circ}{t}), a(\overset{\circ}{t}))$ by using coherent deductive rules of the calculus with pdfs.

This combination of information sources is a powerful internally consistent framework describing the essence of learning. Due to its deductive structure, the incorrect modelling not an incorrect information processing can be blamed for a failure of the specific learning process.

2.5.2 Bayesian estimation

This section deals with a special version of filtering called *estimation*. It arises when the internal quantities Θ_t are time invariant $\Theta_t = \Theta, \forall t \in t^*$. The common value Θ is called *parameter*.

In this case, the time evolution model is $f(\Theta_t|\Theta_{t-1}, a_t, \mathcal{P}_{a_t}) = \delta(\Theta_t - \Theta_{t-1})$, where the employed Dirac delta function $\delta(\cdot)$ is a formal pdf of the measure fully concentrated on the zero argument.

The parametrized model, generally defined in (2.34), has the form

$$\{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta)\}_{t \in t^*}. \quad (2.41)$$

Proposition 2.5.2.1 (Generalized Bayesian estimation) *Let Requirement 2.5.1.1 be met with time invariant $\Theta_t = \Theta \in \Theta^* \subset \mathcal{F}_{a_\tau}, \forall \tau \in t^*$. Then, the outer model of the system (2.20) is given by the formula*

$$f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}) = \int_{\Theta^*} f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta) f(\Theta|\mathcal{P}_{a_t}) d\Theta. \quad (2.42)$$

The evolution of the pdf $f(\Theta|\mathcal{P}_{a_t})$, called (*generalized Bayesian*) parameter estimation generating parameter estimate coinciding with the posterior pdf of unknown parameter, is described by the

recursion identical with data updating (2.39)

$$f(\Theta|\mathcal{P}_{a_{t+1}}) = \frac{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta)f(\Theta|\mathcal{P}_{a_t})}{f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})} \propto f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta)f(\Theta|\mathcal{P}_{a_t}). \quad (2.43)$$

It starts from the prior pdf $f(\Theta) \equiv f(\Theta|\mathcal{P}_{a_1}, a_1) = f(\Theta|\mathcal{P}_{a_1})$.

The simplicity of the estimation formula allows us to write down its (non-recursive) batch variant

$$f(\Theta|\mathcal{P}_{a_{t+1}}) = \frac{\prod_{\tau \leq t} f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)f(\Theta)}{\int_{\Theta^*} \prod_{\tau \leq t} f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)f(\Theta) d\Theta} \equiv \frac{\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}})f(\Theta)}{\mathcal{I}(\mathcal{P}_{a_{t+1}})}. \quad (2.44)$$

The introduced likelihood function

$$\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}}) \equiv \prod_{\tau \leq t} f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta) \quad (2.45)$$

evolves according to the recursion identical with that for the posterior pdf (2.43) but it starts from the $\mathcal{L}(\Theta, \mathcal{P}_{a_1})$ identically equal to 1.

The normalization factor $\mathcal{I}(\cdot)$ is defined by the formula

$$\mathcal{I}(\mathcal{P}_{a_{t+1}}) = \int \mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}})f(\Theta) d\Theta \propto f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}). \quad (2.46)$$

With it, the outer model of the system (2.20) can alternatively be expressed

$$f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}) = \frac{\mathcal{I}(\mathcal{P}_{a_{t+1}})}{\mathcal{I}(\mathcal{P}_{a_t})}. \quad (2.47)$$

Proof: It is again a simple exercise in calculus with pdfs, marginalization, chain rule and Bayes rule, Proposition 2.3.2.2 with the natural conditions of decision-making (2.36). \diamond

Remark(s) 2.5.2.2

1. The observation model $f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta)$ is called parameterized model whenever estimation problem is considered. We respect this tradition.
2. Note that the recursive evolution of the pdf $f(\Theta|\mathcal{P}_{a_t})$ allows us to interpret the posterior pdf as the prior one before processing new observations.
3. The data inserted into the “objective” parameterized (observation) model correct gradually the subjectively chosen prior pdf $f(\Theta)$. The posterior pdf $f(\Theta|\mathcal{P}_{a_t})$ reflects always both ingredients. If the data are informative enough, the relative contribution of the single subjective factor $f(\Theta)$ to the posterior pdf is decreasing with increasing t as the likelihood function $\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}})$ contains t “objective” factors, cf. (2.45).
4. Zero values are preserved by multiplication. Thus, the posterior pdf re-distributes probability mass only within the support of the prior pdf, i.e. within the set

$$\text{supp}[f(\Theta)] \equiv \{\Theta \in \Theta^* : f(\Theta) > 0\}.$$

This fact allows us to introduce hard bounds on possible parameter values but prevents us to “learn” about parameters Θ out of the support $\text{supp}[f(\Theta)]$, see (2.44).

5. Remarks 2.5.1.5 related to filtering apply mostly to estimation, too. The parameter estimation is a task on its own; unknown parameters are always in the ignorance of the decision to be chosen; under the natural conditions of decision-making (2.36), decisions are needed and not the strategy
 $\{\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow a_t^*\}_{t \in t^*}$ generating them.
6. The parameters Θ_t are usually assumed to be finite-dimensional in order to avoid technicalities related to measure theory. In exceptional cases, like in description of so called equivalence approach, see Section 4.2.1, we deal with potentially infinite-dimensional parameter. It means that the number of unknown quantities is finite but increases without limitations. This case is often called non-parametric estimation.

2.5.3 Asymptotic of estimation

The analysis outlined here serves us mainly for interpretation purposes. Thus, all technicalities are suppressed as much as possible.

The “objective” pdf $f(\mathcal{Q})$, see Section 2.3.1, describing the system behavior is denoted here $^{[o]}f(\mathcal{Q})$. The corresponding outer model of the system $f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})$ is denoted $^{[o]}f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})$. Its relationship to the predictive pdf $f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})$ obtained through the parameter estimation, Proposition 2.5.2.1, is inspected.

For the analysis, the notion of *entropy rate* $\mathcal{H}_\infty \left(^{[o]}f || \Theta \right)$ is needed. For each $\Theta \in \Theta^*$, it is defined by the formula

$$\begin{aligned} \mathcal{H}_\infty \left(^{[o]}f || \Theta \right) &\equiv \limsup_{t \rightarrow \infty} \mathcal{H}_t \left(^{[o]}f || \Theta \right) \equiv \\ &\equiv \lim_{t \rightarrow \infty} \sup \frac{1}{t} \sum_{\tau \leq t} \int_{\Delta_{a_\tau}^*} ^{[o]}f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}) \ln \left(\frac{^{[o]}f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})} \right) d\Delta_{a_\tau}. \end{aligned} \quad (2.48)$$

Proposition 2.5.3.1 (Asymptotic of estimation) *Let the natural conditions of decision-making (2.36) hold and $0 < \underline{C}_\Theta \leq \bar{C}_\Theta \leq c < \infty$, $\bar{t}_\Theta \in \{1, 2, \dots\}$ exist, for almost all $\Theta \in \Theta^*$, such that*

$$\underline{C}_\Theta f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta) \leq ^{[o]}f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}) \leq \bar{C}_\Theta f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta), \quad \forall t > \bar{t}_\Theta, \quad \forall \mathcal{P}_{a_{t+1}} \in \mathcal{P}_{a_{t+1}}^*. \quad (2.49)$$

Then, the posterior pdf $f(\Theta|\mathcal{P}_{a_t})$ (2.43) converges almost surely to a pdf $f(\Theta|\mathcal{P}_{a_\infty})$. It has the support coinciding with the set of minimizing arguments in

$$\text{supp} [f(\Theta|\mathcal{P}_{a_\infty})] = \text{Arg} \min_{\Theta \in \text{supp}[f(\Theta)] \cap \Theta^*} \mathcal{H}_\infty \left(^{[o]}f || \Theta \right). \quad (2.50)$$

Proof: Under the natural conditions of decision-making (2.36), the posterior pdf (2.43) can be written in the form

$$f(\Theta|\mathcal{P}_{a_{t+1}}) \propto f(\Theta) \exp [-t\mathcal{H}(\mathcal{P}_{a_{t+1}}, \Theta)] \quad \text{with} \quad (2.51)$$

$$\mathcal{H}(\mathcal{P}_{a_{t+1}}, \Theta) = \frac{1}{t} \sum_{\tau \leq t} \ln \left[\frac{^{[o]}f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)} \right]. \quad (2.52)$$

This form exploits the fact that non-normalized posterior pdf can be multiplied by any factor independent of Θ .

Let us fix the argument $\Theta \in \Theta^*$ and define

$$\begin{aligned} e_{\Theta;\tau} &\equiv \ln \left[\frac{[{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)} \right] - [{}^o]\mathcal{E} \left[\ln \left[\frac{[{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)} \right] \middle| a_\tau, \mathcal{P}_{a_\tau} \right] \\ &\equiv \ln \left[\frac{[{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)} \right] - \int [{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}) \ln \left[\frac{[{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)} \right] d\Delta_{a_\tau}. \end{aligned}$$

A direct check reveals that the introduced deviations $e_{\Theta;\tau}$ of values

$$\ln \left[\frac{[{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})}{f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau}, \Theta)} \right]$$

from their conditional expectations $[{}^o]\mathcal{E}[\cdot|a_\tau, \mathcal{P}_{a_\tau}]$, given by $[{}^o]f(\Delta_{a_\tau}|a_\tau, \mathcal{P}_{a_\tau})$, are zero mean and mutually non-correlated. With them,

$$\mathcal{H}(\mathcal{P}_{a_{t+1}}, \Theta) = \mathcal{H}_t \left([{}^o]f || \Theta \right) + \frac{1}{t} \sum_{\tau \leq t} e_{\Theta;\tau}.$$

The assumption (2.49) implies that the variance of $e_{\Theta;\tau}$ is bounded. Consequently, the last term in the above expression converges to zero almost surely (a.s.), see [21], page 417. The first term on the right hand side of the last equality is non-negative as it can be viewed as a sum of Kullback-Leibler distances, see Proposition 2.4.2.2. Due to (2.49), it is also finite. Thus, (2.52) converges a.s. to the non-negative value $\mathcal{H} \left([{}^o]f || \Theta \right)$. The posterior pdf remains unchanged if we subtract $t \min_{\Theta \in \text{supp}[f(\Theta)] \cap \Theta^*} \mathcal{H}_\infty \left([{}^o]f || \Theta \right)$ from the exponent of its \mathcal{H} -normalized version (2.51). Then, the exponent contains $(-t \times \text{an_asymptotically_non-negative_factor})$. Thus, the posterior pdf $f(\Theta|\mathcal{P}_{a_\infty})$ may be asymptotically non-zero on minimizing arguments (2.50) only. \diamond

Remark(s) 2.5.3.2

1. *The entropy rate can be seen as an extension of the Kullback-Leibler distance (2.26) that covers well asymptotic and controlled cases. It coincides with the Kullback-Leibler distance in a range of particular cases.*
2. *The assumption (2.49) can be weakened. It is, however, intuitively well acceptable. It excludes parameterized models, which assign zero belief to data generated by the system with a non-zero probability and vice versa.*
3. *The Bayesian estimation minimizes asymptotically the entropy rate of the objective pdf $[{}^o]f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})$ and a model chosen among candidates $f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta)$, $\Theta \in \Theta^*$. In other words, a best projection of the objective pdf to the considered parameterized models is asymptotically found. The prior pdf can be interpreted as a prior belief assigned to the individual parameters $\Theta \in \Theta^*$ that the corresponding parameterized model is the best projection of the objective pdf [22]: not knowing the reality we do not know the best projection we finally arrive at.*
4. *The posterior pdf concentrates on a point if there is a unique minimizer of the entropy rate. In this case, the model is called identifiable. The possibility to identify the model can be influenced by*

- the considered class of the parameterized models and
 - the decisions chosen, for instance, by the controller used: e.g. controller generating constant inputs does not allow us to learn their dynamic influence on outputs.
5. If the objective pdf ${}^{[o]}f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t})$ coincides with $f(\Delta_{a_t}|a_t, \mathcal{P}_{a_t}, \Theta)$ for some $\Theta = {}^{[o]}\Theta$ with $f({}^{[o]}\Theta) > 0$ then ${}^{[o]}\Theta$ is in the support of the asymptotic posterior pdf $f(\Theta|\mathcal{P}_{a_\infty})$. If, moreover, the model is identifiable the objective pdf is asymptotically identified by the adopted Bayesian approach. This fact can be expressed in a more appealing form:

Bayesian estimate is consistent whenever there is a consistent estimator.

6. Often, a similar analysis is performed by measuring distance of parameterized models to empirical pdf of data [23]. It gives similar answers if the empirical pdf converges to the objective pdf. Moreover, it provides hints how to approximate the posterior pdf [24], see also Section 4.2.1. On the other hand, the known conditions of such convergence are more restrictive. For instance, an analysis of the controlled case is much harder.

Problem 2.5.3.3 (How to unify statistics?) *Asymptotic analysis and finite-data oriented Bayesian approach are often perceived in an antagonistic way. Their simultaneous and harmonized use still waits for its full exploitation.*

2.5.4 Exponential family

The exponential family of distribution describing the parametrized model plays decisive role in all decision making tasks. The reason is, that all functional recursions solving the dynamic decision making tasks change into algebraic ones for their statistics. Thus, the tasks are significantly easier to solve within this class then outside it.

Agreement 2.5.4.1 (Exponential family of parameterized models) *The parameterized model belongs to the (dynamic) exponential family iff it can be written in the form*

$$f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) = A(\Theta) \exp[B'(\Psi_t)C(\Theta)], \text{ where} \quad (2.53)$$

$\Psi_t' \equiv [\Delta_{a_t}', \psi_t']$ is a finite dimensional data vector determined by experience $\mathcal{P}_{a_{t+1}}$ and its values can be recursively updated according to a known formula $(\Psi_{t-1}, a_t, \Delta_{a_t})^* \rightarrow \Psi_t^*$,

' means transposition.

$A(\cdot)$ is a non-negative scalar function defined on Θ^* ,

$B(\cdot), C(\cdot)$ are vector functions of the same finite fixed dimension. They are defined on respective arguments in Ψ_t^* and Θ^* .

Remark(s) 2.5.4.2

1. The ψ part of the data vector Ψ is often called regression vector.
2. Our definition of the exponential family is non-standard. The additional requirement on the recursive updating of the data vector Ψ (2.53) is important for the dynamic situations we deal with.

3. Notice that equality is used in (2.53). The normalization of this pdf must not spoil it. Due to it, the allowed form is rather restrictive. Let us indicate it.

Often, it is possible to assume that $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) = \Phi(\Psi_t, \Theta) > 0$. Then, a first order Taylor-type expansion of logarithm of this function implies that $\ln(\Phi(\Psi_t, \Theta)) \approx \tilde{B}'(\Psi_t)\tilde{C}(\Theta)$. We exponentiate the right hand of this approximate equality and try to normalise it so that integral over Δ_{a_t} equals to one. Mostly, we find that the normalising factor $A(\cdot)$ will depend both on entries of ψ_t and Θ . Thus, the resulting pdf lies out of the exponential family.

Practical significance of the exponential family becomes obvious when we specialize to it Proposition 2.5.2.1 describing estimation and prediction.

Proposition 2.5.4.3 (Estimation and prediction in exponential family) *Let natural conditions of decision making, Requirement 2.5.1.1, be met with time-invariant $\Theta_t = \Theta \in \Theta^*$. Let the parameterized model belong to exponential family (2.53). Then, the predictive pdf (the outer model of the system) is given by the formula*

$$f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t) = \frac{\mathcal{I}(V_{t-1} + B(\Psi_t), \nu_{t-1} + 1)}{\mathcal{I}(V_{t-1}, \nu_{t-1})}, \text{ with} \quad (2.54)$$

$$V_t = V_{t-1} + B(\Psi_t), V_0 = 0; \quad \nu_t = \nu_{t-1} + 1, \nu_0 = 0, \text{ and} \quad (2.55)$$

$$\mathcal{I}(V, \nu) = \int_{\Theta \in \Theta^*} A^\nu(\Theta) \exp[V'C(\Theta)] f(\Theta) d\Theta \quad (2.56)$$

where $f(\Theta)$ is a prior pdf. The Bayesian parameter estimate (posterior pdf) is

$$f(\Theta|\mathcal{P}_{a_{t+1}}) = \frac{A^{\nu_t}(\Theta) \exp[V_t'C(\Theta)] f(\Theta)}{\mathcal{I}(V_t, \nu_t)}, \quad (2.57)$$

i.e. the likelihood function is $\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}}) \equiv \mathcal{L}(\Theta, V_t, \nu_t) = A^{\nu_t}(\Theta) \exp[V_t'C(\Theta)]$.

Remark(s) 2.5.4.4

1. Estimation and prediction within exponential family is extremely simple. The problem of updating of functions (pdfs) converts into the algebraic recursive updating of the finite-dimensional sufficient statistics V_t and of the sample counter ν_t . The wish to have this recursion complete explains the requirement for possibility to update Ψ_t recursively, see Agreement 2.5.4.1. Moreover, a single type of the normalization integral $\mathcal{I}(V, \nu)$ has to be evaluated.
2. An inspection whether there is a wider set of parameterized models with advantageous properties of the exponential family opens just narrow space [25, 26]. Essentially, the exponential family coincides with all parameterized models that are sufficiently smooth functions of Θ and with supports independent of Θ . Uniform distribution with unknown constant boundaries represents one of a few feasible examples of pdfs out of exponential family, see Chapter ??.
3. The class of models that lead to a finite-dimensional characterization of pdfs occurring in filtering is even more restrictive. Its discussion can be found in [27].

Chapter 3

Decision making tasks

Here, we provide practical comments to elements occurring in all decision tasks. Then we list usual decision tasks met. Their specific cases are in the second part of this text. Their presentation follows more or less the structure of this chapter.

3.1 Construction elements

The discussion of “atoms” creating any decision task should help us to learn a good practice and to avoid common mistakes. Their choice, modification and use represent typically specific decision subtasks that have to be harmonized with the final aim considered. It is often very hard task but the following *design golden rule* should be respected as much as possible:

$$\text{Do not split the overall decision task into subtasks unless necessary.} \quad (3.1)$$

3.1.1 Data

Data connect the artificial world of evaluations with reality. Their information content is crucial for the success of the decision making that use them.

Experimental design

If the designer has an opportunity to influence the information content of data he/she should do that by a proper *experimental design*. At abstract level, it means that the chosen working conditions should suppress ambiguity of the best projections caused by quality of data, see Proposition ???. More practically, the optional data (inputs, set-points) have to be chosen so that they “excite” the system in a sufficient way. For instance, we cannot learn the dependence of outputs on inputs when inputs are not varying during the data acquisition. Obviously, data properties influence also speed of the learning.

Thus, if possible, it is reasonable to optimize information content of data. We outline such optimization in the simpler case of estimation.

Mutual information, i.e. Kullback-Leibler distance (2.26) of the joint pdf of data and parameters to the product of their marginal pdfs, is known to be an adequate quantitative expression of the informational relationship inspected [15]. Under natural conditions of decision making

(2.36), it takes the form

$$\begin{aligned} \mathcal{I}(\Delta(\hat{t}), a(\hat{t})|\Theta) &\equiv & (3.2) \\ &\int f(\Delta(\hat{t}), a(\hat{t}), \Theta) \ln \left(\frac{f(\Delta(\hat{t}), a(\hat{t}), \Theta)}{f(\Delta(\hat{t}), a(\hat{t}))f(\Theta)} \right) d(\Delta(\hat{t}), a(\hat{t}), \Theta) = \sum_{t \in t^*} \\ &\int f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) f(a_t|\mathcal{P}_{a_t}) f(\Theta|\mathcal{P}_{a_t}) f(\mathcal{P}_{a_t}) \ln \left(\frac{f(\Theta|\mathcal{P}_{a_t})}{f(\Theta)} \right) d(\Delta(t), a(t), \Theta). \end{aligned}$$

This form demonstrates that the mutual information measures an average distance of the posterior and prior pdfs.

In order to learn as much as possible about unknown parameters, the mutual information should be maximized with respect to the optional admissible strategy described by its model $\{f(a_t|\mathcal{P}_{a_t})\}_{t \in t^*}$. Formally, dynamic programming, Proposition 2.4.1.1, could be used to this purpose.

Remark(s) 3.1.1.1

1. *The functional (3.2) depends linearly on pdfs $\{f(a_t|\mathcal{P}_{a_t})\}_{t \in t^*}$ describing the optional strategy. Thus, a practically meaningful solution can be obtained only when some restrictions are added. This property is not related to this specific distance but manifests the fact that an infinite stimulating energy, if allowable, may provide the best information about the system inspected.*
2. *The formulated maximization task cannot be generally solved because of its complexity. It is as complex as dual control, see Section ???. It provides us at least the ideal solution to be approximated. For instance, it becomes a useful optimizing guide when only a finite amount of competitive strategies is a priori allowed.*

Mutual information can also be used for analyzing data when results of parameter estimation are unsatisfactory.

3. *Specific feasible solutions for specific classes of models are well elaborated see, for instance, [28].*

Problem 3.1.1.2 (How to approximate experimental design?) *General, numerically feasible approximations of the optimal but infeasible solution of the experimental design are still waiting for their inventors. The problem is even more hard and challenging when the predictive pdfs result from filtering, see Section 2.5.1. It might be reasonable to consider the asymptotic version of this optimization like in Section 2.4.3.*

Data pre-processing

The objective pdf describing reality is (practically) always out of the set considered parameterized models. The estimation searches for the best projection of all aspects of reality which are reflected in measured data, cf. Remarks ???. Some aspects may be a priori uninteresting for the problem at hands. The estimation, that always deals with a finite data set, has to handle them, too. Consequently, it might fail to provide a solid information on aspects of interest. Thus, a wise and commonly used practice is to suppress uninteresting details in data before estimation

or filtering, to pre-process data. A similar logic justifies pre-processing in control and other decision making tasks.

Pre-processing steps could be and ideally should be done using the framework described in this text. Some of them form essence of the presented examples, see, for instance, Sections ???. At the same time, there is a bunch of ready technique [29, 30] that can be *carefully* used.

Typical pre-processing steps are: data transformation linearizing their relationships, data scaling, outliers removal, high frequency noise removal, filling of missing data, data aggregation etc. Each of them makes a pre-projection to a simplified world. It obviously influences quality of the resulting projection. Damages made in the pre-processing phase can hardly be removed in later design phases. Typical errors in pre-processing are:

- a premature reduction of data so that some informative attributes are lost,
- wasting of information in data by sampling them with too low frequency,
- a significant change of the modelled dynamics by underestimating the fact that the the final model describes coupling of the reality and the used pre-processing mechanism,
- a distortion of the inspected relationships by inadequate substitutions for missing data.

Problem 3.1.1.3 (How to harmonize pre-processing with ultimate goal?) *Similarly, as in other places, the optimal pre-processing demands to solve the decision problem in its entirety. It is mostly impossible and splitting the overall task into adequate and harmonized subtasks is left to a “sound” reasoning. It is pleasant as it requires creativity. It is unpleasant as the overall results of the decision making might be spoiled by an improper choice. The problem is severe especially in dynamic decision making problems in which there is a restricted freedom for an iterative trial-and-error treatment.*

3.1.2 Decisions

Elements

The set a^* of possible decisions contains often elements that can “substitute” each other. This incomplete pre-determination calls for selecting the most suitable elements. For instance, moisture of a produced paper can be influenced both by machine speed and energy used for drying [31]. The choice is first of all based on physical and economic priorities. Then structure estimation task, see Agreement 3.3.3.2 and Proposition 3.3.3.3, should follow when no other technical priorities can be recognisee.

Restrictions

Causality restrictions, see Section 2.2, are always considered in our design. Their impact can be influenced at a pre-design stage. Typically, new sensors can be installed in order to enrich accessible experience. Their efficient choice is far from being trivial and respecting of the design golden rule (3.1) cannot be over-stressed in this instance.

Technological restrictions, i.e. final specification of the decision set $a^*(\hat{t})$, follows mostly from technological, economic or safety considerations. Generally, they increase complexity of the design of the optimal strategy and have to be modified in order to make the design and application feasible. When they are relaxed for complexity reasons, they have to be reflected in other

elements of the design. Often, the loss function is modified. In a sense, a sort of penalty-based minimization under restrictions is implicitly used.

Moreover, a good and justified optimization practice has to be respected. For instance, when a particular unrestricted optimizing decision is out of the target set a^* a simple clipping at the boundary of a^* should not be used: a proper (near) optimal projection on a^* is needed.

Also, unnecessary restrictions should be avoided. For instance, the restriction to unbiased estimators [32] is well justified in many statistical decision tasks. If it is, however, mechanically required for other tasks, say adaptive control design, it can make the final strategy (much) less efficient than possible.

Problem 3.1.2.1 (How to treat technical priorities?) *A systematic analysis of priorities and technical consideration mentioned in previous paragraphs and met on many places should be supported by formal tools. Those available now, like qualitative modelling and simulation [33], do not fit well the methodology described here.*

3.1.3 Parameterized model

Again, a non-formalized search for a compromise between the model quality and complexity of its treatment makes the model choice more art than science.

Exponential family

The need for reaching this compromise is responsible for a prominent role of so called *exponential family* [34] and *conjugate prior*. They help us in struggle with the curse of dimensionality [35] and at the same time they are able to cover a range of practical problems.

Majority of the textbook pdfs belong to the exponential family. The following table provides examples of such pdfs in the *static case* when the regression vector is empty, i.e. $\Psi \equiv \Delta_a \equiv y$ and y is either scalar or its entries are independent (recall Agreement 2.3.2.1).

Name	Parameterized model	Innovation	Parameter
Exponential	$\frac{1}{\lambda} \exp(-\frac{y}{\lambda})$	$y \geq 0$	$\lambda > 0$
Poisson	$\frac{\mu^y}{\Gamma(y+1)} \exp(-\mu)$	$y \in \{0, 1, \dots\}$	$\mu > 0$
Multinomial	$\prod_{i \in y^*} \Theta_i^{\delta(i,y)}$	$y \in \{1, \dots, \hat{y}\}$	$\{\Theta_i \geq 0, \sum_{i \in y^*} \Theta_i = 1\}$
Normal	$\frac{1}{\sqrt{2\pi r}} \exp\left[-\frac{(y-\mu)^2}{2r}\right]$	$y \in (-\infty, \infty)$	$\mu \in (-\infty, \infty), r > 0$
Log-normal	$\frac{1}{y\sqrt{2\pi r}} \exp\left[-\frac{\ln^2(\frac{y}{\mu})}{2r}\right]$	$y > 0$	$\mu > 0, r > 0$

The Euler gamma function Γ (??) and Kronecker delta function δ (??) are defined in Chapter ??.

The *dynamic case* with a nonempty ψ is much more narrow (cf. Remarks above). Essentially, normal (Gaussian) linear-in-parameters model and Markov chains dominate this case.

Name	Parameterized model	Data vector	Parameter
Normal (Gaussian)	$\frac{1}{\sqrt{2\pi r}} \exp\left[-\frac{(y-\theta^*\psi)^2}{2r}\right]$	$y \in (-\infty, \infty)$ $\psi \in (-\infty, \infty)^{\hat{\psi}}$	$\theta \in \theta^* \subset (-\infty, \infty)^{\hat{\theta}}$ $r > 0$
Markov chain	$\prod_{i \in y^*} \prod_{j \in \psi^*} \Theta_{i j}^{\delta(i,y)\delta(j,\psi)}$	$y \in y^* \equiv \{1, \dots, \hat{y}\}$ $\psi \in \psi^* \equiv \{1, \dots, \hat{\psi}\}$	$\forall j \in \psi^*, \{\Theta_{i j} \geq 0,$ $\sum_{i \in y^*} \Theta_i = 1\}$

Completion of probabilistic models

The presented design relies (with a few exceptions) on a complete specification of the parameterized model. At the same time, the designer has always at disposal a finite amount of facts for its construction. Their necessary *completion to pdf* (an infinite-dimensional object) represents a non-standard and weakly supported interpolation task. We sketch its solution proposed in [36].

Let the designer know the form of the conditional expectation of a vector function $H(\Delta_{a_t})$, i.e.

$$\mathcal{E}[H(\Delta_{a_t}, \mathcal{P}_{a_t}, a_t)|\Theta, \mathcal{P}_{a_t}, a_t] = h(\Theta, \mathcal{P}_{a_t}, a_t) \quad (3.3)$$

with a known vector function h . The constructed pdf $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)$ should respect just this knowledge. It has to fulfill (3.3). It should be as uncertain as possible in order to leave as much freedom as possible for learning from data. We quantify these requirements by minimizing Kullback-Leibler distance (2.26) of the constructed $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)$ to the uniform pdf under the restriction (3.3). Note that the minimization is equivalent to entropy maximization [15].

Proposition 3.1.3.1 (Completion of parameterized models) *The minimizer $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)$ of the functional*

$$\mathcal{D}(f||\text{uniform pdf}) = \int f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) \ln[f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)] d\Delta_{a_t}$$

within the set of pdfs $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) \in$

$$\left\{ \int_{\Delta_{a_t} \in \Delta_{a_t}^*} f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) d\Delta_{a_t} = 1, \mathcal{E}[H(\Delta_{a_t}, \mathcal{P}_{a_t}, a_t)|\Theta, \mathcal{P}_{a_t}, a_t] = h(\Theta, \mathcal{P}_{a_t}, a_t) \right\}$$

has the form

$$f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) = \gamma(\Theta, \mathcal{P}_{a_t}, a_t) \exp[\lambda'(\Theta, \mathcal{P}_{a_t}, a_t)H(\Delta_{a_t}, \mathcal{P}_{a_t}, a_t)]. \quad (3.4)$$

The scalar function $\gamma(\cdot)$ is determined by the normalization, see Proposition 2.3.2.2. The vector function $\lambda(\cdot)$ has to fulfill the identity

$$\int_{\Delta_{a_t}^*} H(\Delta_{a_t}, \mathcal{P}_{a_t}, a_t) \gamma(\Theta, \mathcal{P}_{a_t}, a_t) \exp[\lambda'(\Theta, \mathcal{P}_{a_t}, a_t)H(\Delta_{a_t})] d\Delta_{a_t} = h(\Theta, \mathcal{P}_{a_t}, a_t). \quad (3.5)$$

Proof: It is implied by an elementary calculus of variations. ◇

Remark(s) 3.1.3.2

1. The result is more of a conceptual than of a practical interest as the solution of (3.5) can rarely be found. Moreover, the model found is mostly out of the favorite exponential family.
2. Gaussian parameterized model is obtained for $H(\Delta_{a_t}, \mathcal{P}_{a_t}, a_t) = [\Delta_{a_t}, \Delta_{a_t}^2]'$. This helps us in judging suitability of this popular option.

Problem 3.1.3.3 (How to make the completion practicable?) *The weakness of the presented results indicates how neglected the art of the model completion is. The completion has to be either developed much more or avoided completely. The latter desirable direction would, however, require to build the corresponding decision theory relying only on partial descriptions like (3.3). It is possible that it is impossible.*

Mixtures of models

The model constructed in previous Section exploits available physical knowledge expressed by the expectation (3.3). It is often obtained through modelling valid locally around some working point. Then, an extension to models covering a wider range of working conditions is desirable.

Mixture of models

$$f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) = \sum_{c \in c^*} \alpha_c f_c(\Delta_{a_t}|\Theta_c, \mathcal{P}_{a_t}, a_t), \quad \Theta \in \Theta^* \equiv \left\{ \Theta = (\alpha_c, \Theta_c), \Theta_c \in \Theta_c^* \quad c \in c^* \equiv \{1, \dots, \dot{c} < \infty\}, \alpha \in \alpha^* \equiv \left\{ \alpha_c \geq 0, \sum_{c \in c^*} \alpha_c = 1 \right\} \right\} \quad (3.6)$$

represents a good universal option for meeting this need as the mixtures of models have often so called *universal approximation property*. It means that the set (3.6) is dense in the set of outer models of the system. This property takes place whenever the number of *components* \dot{c} is not limited and individual components, i.e. pdfs $f_c(\Delta_{a_t}|\Theta_c, \mathcal{P}_{a_t}, a_t)$, can be made arbitrarily close to an arbitrarily positioned Dirac delta function [37].

The *exact estimation and prediction* with this model is practically *impossible* as the number of terms in likelihood function $\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}})$ – the product of parameterized models – increases exponentially. Luckily enough, good algorithms for approximate estimation of the mixture of models are available [38]. Even a good approximate *recursive* estimation is possible if the *components* $\{f_c(\Delta_{a_t}|\Theta_c, \mathcal{P}_{a_t}, a_t)\}_{c \in c^*}$ belong to the exponential family or describe uniform distributions on rectangular boxes. This quasi-Bayes estimation is described in Section 4.2.3.

The quasi-Bayes estimation as well as other algorithms exploit the following alternative interpretation of the mixture of models (3.6). Let $c_t \in c^*$ be the unobservable random selector of the “active” component that takes the value c with probability α_c . Then,

$$f(\Delta_{a_t}, c_t|\Theta, \mathcal{P}_{a_t}, a_t) = f_{c_t}(\Delta_{a_t}|\Theta_{c_t}, \mathcal{P}_{a_t}, a_t)\alpha_{c_t}. \quad (3.7)$$

The marginalisation rule, see Proposition 2.3.2.2, implies that the mixture (3.6) is marginal pdf of the joint pdf (3.7).

3.1.4 Prior pdf

The need for selecting prior pdf is often regarded as the main disadvantage of the adopted Bayesian approach []. We contribute positively to the never-ending discussion on its pros and cons by showing that its use opens a regular gate through which prior “expert” information can be introduced into learning in a systematic way.

The posterior pdf (2.44) is a product of the likelihood function consisting of t factors $f(\Delta_{a_\tau}|\Theta, \mathcal{P}_{a_\tau}, a_\tau)$ and of a single prior pdf $f(\Theta)$. If t is high enough and data bring a sufficient information on Θ then the posterior pdfs obtained for various prior pdfs resemble each other: the role of prior pdf is weak.

The posterior pdf is significantly influenced by the prior pdf when some of the above conditions is not fulfilled. Let us formulate simple Propositions that formalize this.

Proposition 3.1.4.1 (Role of the prior pdf)

1. *Parameter values $\Theta \notin \text{supp}[f(\Theta)]$, for which the prior pdf is zero, get the zero posterior pdf, too. Formally,*

$$\text{supp}[f(\Theta|\mathcal{P}_{a_{t+1}})] = \text{supp}[\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}})] \cap \text{supp}[f(\Theta)].$$

2. *The recursive evolution of the likelihood function*

$$\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}}) = f(\Delta_{a_t} | \Theta, \mathcal{P}_{a_t}, a_t) \mathcal{L}(\Theta, \mathcal{P}_{a_t}), \quad t \in t^*, \quad \mathcal{L}(\Theta, \mathcal{P}_{a_1}) = 1, \Theta \in \Theta_{\mathcal{L}}^* \quad (3.8)$$

does not depend on the prior pdf chosen.

3. *The posterior pdf exists iff the product $\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}})f(\Theta)$ is integrable.*

Proof: It is a direct consequence of the formula for posterior pdf (2.44). \diamond

Remark(s) 3.1.4.2

1. *The prior pdf offers a simple and clear way for introducing hard restrictions on parameters.*
2. *The recursion (3.8) is valid even if $\Theta_{\mathcal{L}}^* \neq \Theta^* \equiv \text{supp}[f(\Theta)]$. This trivial statement implies that hard bounds on parameter values must not influence likelihood function, i.e. the “objective” link to data reflecting reality. This fact is easily and repeatedly overlooked in recursive estimation. Instead of restricting the posterior pdf the statistics determining the likelihood are deformed with an adverse effect on the estimation quality.*
3. *Often, a flat prior pdf is chosen in order to model the lack of prior knowledge. Even integrability of the prior pdf is relaxed and the improper prior pdfs $f(\cdot) \geq 0$, $\int f(\Theta) d\Theta = \infty$ are used. For instance, the posterior pdf is proportional to the likelihood function if we allow the prior pdf be improper and equal 1. Then, however, the posterior pdf might be improper, too. Thus, it is more reasonable to select a flat but proper prior pdf as it serves as a regularizing factor.*

Agreement 3.1.4.3 (Conjugate prior pdf) *Let the parameterized model belong to exponential family, see Agreement 2.5.4.1,*

$$f(\Delta_{a_t} | \Theta, \mathcal{P}_{a_t}, a_t) = A(\Theta) \exp[B'(\Psi_t)C(\Theta)].$$

The prior pdf is called conjugate prior pdf if it has the form

$$f(\Theta) \propto A^{\nu_0}(\Theta) \exp[V_0' C(\Theta)] \chi(\Theta). \quad (3.9)$$

It is determined by the indicator χ of the set Θ^ and “prior statistics” V_0, ν_0 .*

Proposition 3.1.4.4 (Estimation with conjugate prior pdf) *Let the parameterized system belong to the exponential family and the conjugate prior pdf is chosen. Then, the posterior pdf have the same functional form as the prior one. Its statistics V_t, ν_t evolve according to (2.55) with zero initial conditions replaced by V_0, ν_0 :*

$$\begin{aligned} f(\Theta | \mathcal{P}_{a_t}) &\propto A^{\nu_t}(\Theta) \exp[V_t' C(\Theta)] \chi(\Theta), \quad t \in t^* \\ V_t &= V_{t-1} + B(\Psi_t), \quad \nu_t = \nu_{t-1} + 1 \\ V_0, \nu_0 &\equiv \text{prior statistics.} \end{aligned} \quad (3.10)$$

Proof: It is a direct consequence of the formula for posterior pdf (2.44) applied to the exponential family. \diamond

Agreement 3.1.4.5 (Fictitious data) *Let the parameterized model belong to the exponential family and a conjugate prior pdf $f(\Theta) \equiv f(\Theta|K_{t_f})$ respects a piece of knowledge K_{t_f} . The fictitious data vector Ψ_{t_f} is said to express this knowledge piece if*

$$f(\Theta|K_{t_f}) \propto A^{\bar{v}+1}(\Theta) \exp\{[B(\Psi_{t_f}) + \bar{V}]'C(\Theta)\} \propto A(\Theta) \exp[B'(\Psi_{t_f})C(\Theta)]\bar{f}(\Theta). \quad (3.11)$$

The used pdf $\bar{f}(\Theta) \propto A^{\bar{v}}(\Theta) \exp[\bar{V}'C(\Theta)]$ is a flat pre-prior pdf.

In other words, the inclusion of the individual knowledge piece K_{t_f} is expressed as an application of the Bayes rule with fictitious data vector inserted into the parameterized model.

Remark(s) 3.1.4.6

1. Instructive examples of fictitious data are in in [39]. The construction is also demonstrated on examples of this text. The common procedure is as follows:

Algorithm 3.1.4.7 (Construction of fictitious data)

- (a) Ask yourselves what response of the system you expect to some experience (often, past inputs or initial values) when your information piece is correct.
 - (b) Think about uncertainty of the system response caused by your uncertainty about the treated knowledge piece.
 - (c) Scale these data so that the response uncertainty resembles the uncertainty caused by random noises within the system.
2. It is possible and sometimes reasonable to group fictitious data into mutually consistent blocks that are treated as a block of real data vectors, i.e. modify the flat prior pdf through a plain rule of Bayes rule.

A collection of knowledge pieces $\{K_{t_f}\}_{t_f \in t_f^*}$ contains generally items of heterogeneous and uncertain nature. The auxiliary notion of fictitious data provides us their unified expression and give us a possibility to reflect their uncertainty. With fictitious data vectors, the knowledge items $K(\hat{t}_f)$, $\hat{t}_f < \infty$ can be incorporated into the prior pdf individually. Without knowing their mutual dependencies, like their complementarity, degree of compatibility, repetitions etc., we cannot construct the desirable prior pdf $f(\Theta|K(t_f))$ reflecting the whole collection of available knowledge pieces. Thus, some heuristics is needed. The following simple proposition prepares one of possible and successful formalizations.

Proposition 3.1.4.8 (Geometric representant of a pair of pdfs) *Let an unknown pdf $f \in f^* \equiv \{f_1, f_2\}$ be equal to $f_1(x)$ with a probability $\alpha \in \alpha^* \equiv [0, 1]$ and equal to $f_2(x)$ with the complementary probability $1 - \alpha$. The pdfs f_1, f_2 are supposed to have a common support x^* . Then, the pdf*

$$^{[0]}\hat{f}(x) \propto f_1^\alpha(x)f_2^{1-\alpha}(x), \quad x \in x^* \quad (3.12)$$

minimizes the expected Kullback-Leibler distance (2.26) of $\hat{f}(\cdot)$ to $f(\cdot)$, i.e. the functional

$$\mathcal{E}[\mathcal{D}(\hat{f}||f)] \equiv \alpha\mathcal{D}(\hat{f}||f_1) + (1 - \alpha)\mathcal{D}(\hat{f}||f_2). \quad (3.13)$$

The minimum reached is

$$\omega(\alpha) \equiv \mathcal{E}[\mathcal{D}(^{[0]}\hat{f}||f)] = -\ln \int_{x^*} f_1^\alpha(x)f_2^{1-\alpha}(x) dx. \quad (3.14)$$

If $f_1 \neq f_2$, the function $\omega(\alpha)$ reaches its maximum on $(0, 1)$.

Proof: (3.12) is implied by an elementary calculus of variations. (3.14) follows from it and (3.13). The last statement is implied by the assumption $f_1 \neq f_2$, closeness of α^* and continuity of $\omega(\alpha)$ guaranteed by the common support of f_1, f_2 . \diamond

The combination or merging of a pair of knowledge pieces may be directly based on this proposition by taking $f_{t_f} \equiv f(\Theta|K_{t_f})$, $t_f = 1, 2$ and accepting $^{[o]}\hat{f}$ as $f(\Theta|K(t_f))$. The constructed prior pdf used should not claim misleading confidence thus it makes sense to choose the “worst” (conservative) α in (3.12), i.e. α that maximizes the minimum reached.

It is straightforward to extend Proposition 3.1.4.8 for several pooled pdfs. However, the search for the worst probabilistic vector α becomes computationally intensive even for a medium t_f .

There is an obvious approximation based on a sequential application of Proposition 3.1.4.8 with $f_1 \equiv f(\Theta|K(t_f - 1))$, $f_2 \equiv f(\Theta|K_{t_f})$ and $^{[o]}\hat{f} \equiv f(\Theta|K(t_f))$. The result may, however, strongly depend on the evaluation order. For this reason, an alternative formulation has been proposed in [39] that allows us not only to add a knowledge piece but also to remove superfluous ones.

Let us describe this procedure assuming that $f(\Theta|K(t_f - 1))$ has been gained in some way and that there are two possibilities to built-in a new piece of knowledge K_{t_f} :

Optimistic variant: Bayes rule includes correctly the new knowledge piece

$$f_1(\Theta|K(t_f)) \propto f(\Delta_{a_{t_f}}|\Theta, \mathcal{P}_{a_{t_f}}, a_{t_f})f(\Theta|K(t_f - 1)).$$

This variant is assigned the probability $\alpha \in [0, 1]$

Cautious variant: Bayes rule can only be applied after removing the new factor $f(\Delta_{a_{t_f}}|\Theta, \mathcal{P}_{a_{t_f}}, a_{t_f})$ β -times

$$f_2(\Theta|K(t_f)) \propto [f(\Delta_{a_{t_f}}|\Theta, \mathcal{P}_{a_{t_f}}, a_{t_f})]^{1-\beta}f(\Theta|K(t_f - 1)), \beta \geq 0.$$

This variant is taken with the probability $1 - \alpha$.

Proposition 3.1.4.9 (Recursive processing of prior knowledge) *Let $f(\Theta|K(t_f))$ has only the optimistic or cautious variant. Then, the minimizer $\hat{f}(\Theta|K(t_f))$ of the expected value of the Kullback-Leibler distance (2.26) to it has the form*

$$\hat{f}(\Theta|K(t_f)) \propto [f(\Delta_{a_{t_f}}|\Theta, \mathcal{P}_{a_{t_f}}, a_{t_f})]^w f(\Theta|K(t_f - 1)), \text{ with } w = \alpha + (1 - \alpha)(1 - \beta).$$

The unique “worst” values of $\alpha \in [0, 1]$ maximizing the reached minimum and

$$\beta = \left\{ \inf \tilde{\beta} : \int [f(\Delta_{a_{t_f}}|\Theta, \mathcal{P}_{a_{t_f}}, a_{t_f})]^{-\tilde{\beta}} f(\Theta|K(t_f - 1)) d\Theta = \infty \right\}$$

exist.

Proof: Essentially, Proposition 3.1.4.8 is extended. For details see [39]. \diamond

In batch processing, when we can combine prior knowledge with the measured data at one shot, it is possible to estimate weights α by using the standard Bayes rule. This straightforward procedure provides directly estimate of the posterior pdf with the available prior knowledge included. We present the result for exponential family.

Agreement 3.1.4.10 (Batch processing of prior knowledge) *Let the parameterized model belong to exponential family (2.53) and prior knowledge pieces $K(\dot{t}_f)$ are expressed through fictitious data, see Agreement 3.1.4.5. We search for such an approximation $\hat{f}(\Theta|\mathcal{P}_{a_t}, K(\dot{t}_f))$ of the unknown posterior pdf $f(\Theta|\mathcal{P}_{a_t}, K(\dot{t}_f))$ that*

- has experience $\mathcal{P}_{a_t}, K(\dot{t}_f)$,
- minimizes the expected Kullback-Leibler distance to projections of $f(\Theta|\mathcal{P}_{a_t}, K(\dot{t}_f))$ within the set

$$\{f(\Theta|\mathcal{P}_{a_t}, K_{t_f})\}_{t_f \in t_f^*} \quad (3.15)$$

- assigns the to the elements of (3.15) the posterior probabilities $\alpha(K_{t_f}|\mathcal{P}_{a_t})$ corresponding to uniform prior probabilities on $K(\dot{t}_f)$.

A straightforward use of Proposition 3.1.4.8, definition of fictitious data, Agreement 3.1.4.5, estimation in exponential family, Proposition 3.1.4.4, and Bayes rule give:

Algorithm 3.1.4.11 (Batch processing of prior knowledge)

1. Compute the statistics V, ν determining the posterior pdf corresponding to the pre-prior pdf only

$$\bar{V}_t = \sum_{\tau \leq t} B(\Psi_\tau) + \bar{V}, \quad \bar{\nu}_t = \bar{\nu} + t \quad (3.16)$$

where the data vectors Ψ_τ are constructed from the (real) data available.

2. Evaluate the posterior probabilities

$$\alpha(t_f|\mathcal{P}_{a_t}) \propto \frac{\mathcal{I}(\bar{V}_t + B(\Psi_{t_f}), \bar{\nu}_t + 1)}{\mathcal{I}(\bar{V} + B(\Psi_{t_f}), \bar{\nu} + 1)}, \quad t_f \in t_f^* \quad (3.17)$$

where the data vectors Ψ_{t_f} are fictitious.

3. Determine statistics of the final pdf $\hat{f}(\Theta|\mathcal{P}_{a_t}, K(\dot{t}_f)) \propto A^{\nu_t}(\Theta) \exp[V'_t C(\Theta)]$

$$V_t = \bar{V}_t + \sum_{t_f \in t_f^*} \alpha(t_f|\mathcal{P}_{a_t}) B(\Psi_{t_f}), \quad \nu_t = \bar{\nu}_t. \quad (3.18)$$

Remark(s) 3.1.4.12

1. The upper bound on the set of possible β values is chosen so that out of it the removal of the factor $f(\Delta_{t_f}|\Theta, \mathcal{P}_{a_{t_f}}, a_{t_f})^{-\beta}$ shifts the product with $f(\Theta|K(t_f - 1))$ out of normalizable pdfs: nothing more can be removed.
2. The combination of pdfs with the worst α is tightly related to so called barycenter of a pdf group, see [40] and its approximation called pooling [41].
3. The results of the batch version do not depend on the order of prior knowledge processing. It is obviously desirable property that favouritize use of this version whenever possible.

Problem 3.1.4.13 (Is the knowledge quantification good enough?) *There is a significant progress in this area but the results are not stabilized yet. For instance, there is little experience with a direct use of Proposition 3.1.4.9 and Algorithm 3.1.4.11. It is worth of being inspected more closely.*

Problem 3.1.4.14 (How to choose version of Kullback-Leibler distance?) *Kullback-Leibler distance is not symmetric in its arguments. Optimization of the alternative version leads to the arithmetic mean of pdfs instead of the geometric one. There are indications [42] that the chosen version should be preferred but a definite answer does not exist.*

3.1.5 Loss function

The *loss function* should express our wishes as precisely as possible. This obvious demand is often violated. Then, the design provides non-acceptable decision strategies. Let us discuss the common mistakes made.

Common mistakes

A wider than the practically admissible set of decisions a^* is often used in order to decrease complexity of the design. Then, the loss function has to penalize values out of the desirable set. Otherwise, the optimal strategy found may generate useless decisions.

Use of loss functions that do not respect decision making as a dynamic process is another common error. Generally, the decision a_t influences whole future of the coupling system \leftrightarrow strategy. It gives unreasonable strategies if this fact is not respected and sequence of decisions is chosen by optimizing a sequence of unrelated decisions, see Section 4.4.1.

Instability of the closed loop with one-step-ahead controllers applied to non-minimum phase systems is the most known demonstration of this property. However, even stable behaviour is achieved with energy wasted. This is a less known manifestation of the same “mistake” [9].

Popular options

A dominating use of a few loss functions reflects an indirect fight with the design complexity. Undoubtedly, the *quadratic loss* is a prominent example of this type. Its general form,

$$\mathcal{Z} = \|\mathcal{Q} - {}^{[I]}\mathcal{Q}\|^2, \quad (3.19)$$

where $\|\cdot\|$ is a quadratic *weighted* norm and ${}^{[I]}$ denotes a given ideal (reference, set-point), has numerous variants. They differ in the target ideal, in the variables involved and weighting used. The quadratic loss:

- + is numerically feasible in connection with linear models,
- + is flexible as it allows us to express priorities by a proper choice of weights,
- + approximates majority of smooth loss functions through a second order Taylor expansion.

It is, however, fair to say that:

- the quadratic loss over-stresses significance of large deviations from the ideal,

- the choice of weights and ideal reflecting true aims is not easy,
- usefulness of the quadratic loss as a “suppressor” of the design complexity decreases sharply whenever the underlying model is not linear or hard restrictions are involved.

When hard bounds on admissible decisions have to be considered, loss functions based on l -one norm might be much better. Also, the design based on l -infinity norm is nowadays widely inspected [43]. It enhances the attention paid to requirements on *robustness* of the designed strategy. Note that robustness is an engineering name to a smooth dependence of the strategy on various elements (initial conditions, model quality, etc.) on which it depends.

Ideal in fully probabilistic design

The choice of the loss function should be harmonized with the model adopted. For instance, if large random deviations are expected then the use of a quadratic loss might be doubtful.

The need for the harmonization is an additional argument for our preference to the fully probabilistic design, see Section 2.4.2. In this case, the loss is determined by the ideal pdf $^{[I]}f(\Delta(\dot{t}), a(\dot{t}))$ we would like to approximate. Generally, it is wise to derive the ideal pdf from the outer description of closed loop $f(\Delta(\dot{t}), a(\dot{t}))$ by replacing those characteristics we hope to influence by their desirable values. In this way, we get, for instance, an equivalent of a well tailored quadratic performance index for linear systems with Gaussian distribution [18]. Then, the choice of adequate weights in the loss is much simplified. We get also a realistic loss if the well modelled system noise is non-normal.

Problem 3.1.5.1 (How to express multiple objectives?) *There are extensive theories how to handle multi-valued loss functions, what properties utility functions should have [44] etc. In spite of this, operational guidelines how to construct the loss function in specific cases are far from being cook-book.*

3.2 Derived elements

3.2.1 Likelihood function

All investigated tasks are based on data measured on the plant we want to decide of. The information extracted from the data measured up to time t (without prior data) is collected in the likelihood function. It is defined as a product of model pdfs with the data substituted which is viewed as a function of the unknown parameter Θ (2.45)

$$\mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}}) \equiv \prod_{\tau \leq t} f(\Delta_{a_\tau} | \Theta, \mathcal{P}_{a_\tau}, a_\tau) \quad (3.20)$$

Remark(s) 3.2.1.1

1. *The form of the likelihood function is often used as a template for the form chosen for the prior pdf. The reason is that reproducibility of the likelihood function when recursively updated, see (3.8).*

3.2.2 The integral \mathcal{I}

This integral is defined in (2.46)

$$\mathcal{I}(\mathcal{P}_{a_{t+1}}) = \int \mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}}) f(\Theta) d\Theta. \quad (3.21)$$

It is frequently uses in the tasks solved.

3.2.3 Posterior pdf

The posterior pdf represents the most general description the unknown parameter Θ . It yields not only point estimates of parameter values in the form of conditional mean (??), but it contains also information about its uncertainty. It can be obtained from likelihood and prior pdf

$$f(\Theta|\mathcal{P}_{a_{t+1}}) \propto \mathcal{L}(\Theta, \mathcal{P}_{a_{t+1}}) f(\Theta) \quad (3.22)$$

with normalization integral $\mathcal{I}(\mathcal{P}_{a_{t+1}})$ (3.21) or recursively

$$f(\Theta|\mathcal{P}_{a_{t+1}}) \propto f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) f(\Theta|\mathcal{P}_{a_t}), \quad (3.23)$$

where in the role of the normalization integral appears the predictive pdf (3.24).

3.2.4 Predictive pdf

It is a distribution of the next data item (in time instant $t + 1$) conditioned on past and present data items but not parameters. It can be computed from model an posterior pdf by integration

$$f(\Delta_{a_{t+1}}|\mathcal{P}_{a_{t+1}}, a_{t+1}) = \int_{\Theta^*} f(\Delta_{a_{t+1}}|\Theta, \mathcal{P}_{a_{t+1}}, a_{t+1}) f(\Theta|\mathcal{P}_{a_t}) d\Theta = \frac{\mathcal{I}(\mathcal{P}_{a_{t+1}})}{\mathcal{I}(\mathcal{P}_{a_t})} \quad (3.24)$$

where the integrals $\mathcal{I}(\mathcal{P}_{a_{t+1}})$ and $\mathcal{I}(\mathcal{P}_{a_t})$ can be found above in (3.21).

Remark(s) 3.2.4.1

1. The pdf (3.24), at time t is the normalization integral for recursive update of posterior pdf (3.23).

3.3 Static tasks

The *static decision making* deals with the design and use of a single decision rule. Proposition 2.3.3.1 describes its optimal design. Specific instances differ in:

- Decomposition of the behaviour $\mathcal{Q} = (\mathcal{P}_a, a, \mathcal{F}_a) \equiv$ (experience, decision, ignorance).
- Admissible rules $\mathcal{R} \in \mathcal{R}^*$ determined by the domain \mathcal{P}_a^* and range a^* .
- Loss function $\mathcal{Z} : \mathcal{Q}^* \equiv (\mathcal{P}_a, a, \mathcal{F}_a)^* \rightarrow [0, \infty]$.

These elements determine the outer model $f(\mathcal{F}_a|\mathcal{P}_a, a)$ needed for a direct application of the basic optimization lemma, Proposition 2.3.3.1.

3.3.1 Point estimation

Point estimation can be cast in the considered framework as follows.

- $\mathcal{Q} \equiv (D, \hat{\Theta}, \Theta) \equiv$
(data at disposal, point estimate, unknown parameter).
- Admissible rules are of the form $\mathcal{R} : D^* \rightarrow \hat{\Theta}^*, \Theta^* \subset \hat{\Theta}^*$
- Loss function \mathcal{Z} measures a distance of $\hat{\Theta}$ and Θ . This distance may depend on data D , too.

The ignorance $\mathcal{F}_{\hat{\Theta}}$ coincides with the unknown parameter Θ . Thus, the outer model of the system needed for decision making is $f(\Theta|D, \hat{\Theta})$. The adopted natural conditions of decision making (2.36) imply that the decision $\hat{\Theta}$ is superfluous in the conditioning. Thus, the needed model coincides with the (generalised) Bayesian estimate $f(\Theta|D)$ (posterior pdf) determined in Proposition 2.5.2.1. For the given D , the optimal point estimate is, Proposition 2.3.3.1,

$$\hat{\Theta}(D) \in \text{Arg} \min_{\hat{\Theta} \in \hat{\Theta}^*} \int \mathcal{Z}(D, \hat{\Theta}, \Theta) f(\Theta|D) d\Theta. \quad (3.25)$$

Remark(s) 3.3.1.1

1. Note that the needed outer model depends on the specific structuring of realizations \mathcal{Q} not on the loss function chosen.

The specific result of the optimization depends of course on the loss function chosen.

2. The quadratic loss function $\mathcal{Z}(D, \hat{\Theta}, \Theta) = \|\Theta - \hat{\Theta}\|_{Q(D)}^2 \equiv \|\Theta\|_{Q(D)}^2 + \|\hat{\Theta}\|_{Q(D)}^2 + 2(\Theta, \hat{\Theta})_{Q(D)}$ is the most popular option. It is determined by a quadratic norm $\|\cdot\|_{Q(D)}$ with a positive definite, data-dependent, kernel $Q(D)$ and the related scalar product $(\cdot, \cdot)_{Q(D)}$. For it, the minimized quantity, see Proposition 2.3.3.1, is quadratic form in $\hat{\Theta}$, namely,

$$\mathcal{E}[\mathcal{Z}|D, \hat{\Theta}] = \mathcal{E}[\|\Theta\|_{Q(D)}^2|D] + \|\hat{\Theta}\|_{Q(D)}^2 + 2(\mathcal{E}[\Theta|D], \hat{\Theta})_{Q(D)} \quad (3.26)$$

$$\Rightarrow \hat{\Theta} \equiv \mathcal{E}[\Theta|D],$$

irrespectively of the kernel $Q(\cdot)$.

This general result looks very promising as just the first moment (expectation) of the posterior pdf is needed. However, its evaluation requires mostly a complete knowledge of $f(\Theta|D)$.

Problem 3.3.1.2 (How to select the ideal pdf for estimation?) *There is little experience how to specify the ideal of the fully probabilistic design 2.4.2 in the case of parameter estimation. The general guideline, see Section 3.1.5 should be followed in this case, too. A systematic inspection is needed.*

3.3.2 Set estimation

Set estimation is tightly related to the point estimation. It can be cast in the considered framework as follows.

- $\mathcal{Q} \equiv (D, \theta^*, \Theta) \equiv (\mathcal{P}_a, a, \mathcal{F}_a) \equiv$
(data, set estimate, unknown parameter).
- Admissible rules are of the form

$$\mathcal{R} : D^* \rightarrow (\theta^*)^* \subset \left\{ \theta^* \subset \Theta^*, \int_{\theta^*} f(\Theta, D) d\Theta \geq \alpha \in (0, 1) \right\} \quad (3.27)$$

where α is a pre-specified probability called *credibility level*.

- Loss function \mathcal{Z} measures volume of the θ^* . The volume specification may depend on data D , too.

Structuring of realizations \mathcal{Q} is similar to the point estimation. This, see Remark 1 in the previous section, implies that the outer system model needed is $f(\Theta|D)$. For the given D , the optimal interval estimate is (Proposition 2.3.3.1)

$$\theta^*(D) \in \text{Arg} \min_{(\theta^*) \in (\theta^*)^*} \int \mathcal{Z}(D, \theta^*, \Theta) f(\Theta|D) d\Theta. \quad (3.28)$$

Remark(s) 3.3.2.1

1. Essentially, α -fractiles of $f(\Theta|D)$ are searched for.
2. Mostly, intervals are taken as the sets considered. Their length is then taken as a measure of their volume. Other choices might be more appropriate when a multivariate Θ is considered. For instance, hyper-ellipsoids fit well to normally distributed parameters.
3. The sets searched for are called *credibility sets* and they are quite close to so called confidence intervals studied in classical statistics.

Problem 3.3.2.2 (Is there a cook-book giving the ideal pdf?) Again, the experience how to specify the ideal of the fully probabilistic design 2.4.2 is missing. The general guideline, see Section 3.1.5 should be followed but a systematic inspection is needed.

3.3.3 Testing of hypothesis

Hypothesis testing, a selection of the best variant among several alternatives, can be put within our framework as follows.

- $\mathcal{Q} \equiv (D, \hat{h}, \{H_h\}_{h \in h^*}) \equiv$
(data at disposal, estimate $\hat{h} \in h^* \equiv \{1, \dots, \hat{h} < \infty\}$ of h , {hypothesis list}).
- Admissible rules are of the form $\mathcal{R} : D^* \rightarrow h^*$.
- Loss function \mathcal{Z} is a (\hat{h}, \check{h}) -table with non-negative entries $\mathcal{Z}(D, \hat{h}, h)$, usually with the zero diagonal as no penalty is paid when $\hat{h} = h$.

The outer system model (2.20) needed is $f(h|D, \hat{h})$. The adopted natural conditions of decision making (2.36) imply that the decision \hat{h} is superfluous in conditioning. Thus, we need $f(h|D)$. This probability function (pf, $\hat{h} < \infty$) is obtained according to Proposition 2.5.2.1 as the generalised Bayesian estimate

$$f(h|D) \propto f(D|h)f(h), \quad h \in h^* \quad (3.29)$$

where $f(h)$ is the prior pf of h 's hypothesis. The pdf $f(D|h)$ relates observed data to individual hypotheses $h \in h^*$. For a given D , the optimal decision in hypothesis testing is, Proposition 2.3.3.1,

$$\hat{h}(D) \in \text{Arg} \min_{\hat{h} \in \hat{h}^*} \sum_{h \in h^*} \mathcal{Z}(D, \hat{h}, h) f(h|D). \quad (3.30)$$

Remark(s) 3.3.3.1

1. Unlike in classical hypothesis testing [32], the testing is performed within a completely specified set of alternatives.
2. If a pair of hypothesis is compared, $\hat{h} = 2$, the decision rule constructed according to the described methodology coincides with a celebrated Neymann-Pearson lemma [1].
In this case, off diagonal elements of $\mathcal{Z}(D, \hat{h}, h)$ penalize the classical errors of the 1st and 2nd kind [32].
3. The needed pdfs are rarely obtained directly. Instead, they are predictive pdfs obtained through filtering or parameter estimation and prediction, see Remarks 2.5.1.5. If other decisions, like control, influence the observed data then factors describing their generators cancels in (3.29). It follows from natural conditions of acting (2.36) that are supposed to be valid.
4. In spite of its formal simplicity, the testing of hypothesis is extremely powerful technique. It is especially true when dealing with predictive pdfs needed when so called compound hypothesis are treated, [32]. It, for instance, brought a whole set of novel and efficient solutions of so called structure estimation problem [45, 46, 47] (see below).
5. When dealing with predictive pdfs, a special care has to be devoted to the choice of the prior pdf of unknown parameters within the respective hypothesis. Their practical influence may be unexpectedly high.

The importance of the structure estimation task calls for its formal specification:

Agreement 3.3.3.2 (Structure estimation) *Let us consider parameterized models $\{f(\Delta_{a_t}|\Theta_h, \mathcal{P}_{a_t}, a_t, h)\}_{h \in h^*}$ as candidates for describing a single system. Then, the hypothesis testing about the best value $h \in h^*$ is called structure estimation. The generator of the additional decisions a_t (like system inputs) is supposed to fulfill natural conditions of decision making (2.36).*

Proposition 3.3.3.3 (Solution of structure estimation) *Let data $D \equiv (\mathcal{P}_{a_{t+1}}, a_{t+1})$ be available for structure estimation. Then, the outer model needed for selecting the best structure estimate according to (3.30) is*

$$\begin{aligned} f(h|\mathcal{P}_{a_{t+1}}) &\propto f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t, h) f(h|\mathcal{P}_{a_t}) \equiv \\ &\equiv \int f(\Delta_{a_t}|\Theta_h, \mathcal{P}_{a_t}, a_t, h) f(\Theta_h|\mathcal{P}_{a_t}) d\Theta_h f(h|\mathcal{P}_{a_t}). \end{aligned} \quad (3.31)$$

The pf $f(h|\mathcal{P}_{a_t}, a_t)$ can be obtained recursively according to the formula (3.31) starting from a given prior pf $f(h) \equiv f(h|\mathcal{P}_{a_1})$. The needed generalised Bayesian parameter estimate evolves according to Bayes rule, $h \in h^*$,

$$f(\Theta_h|\mathcal{P}_{a_t}) \propto f(\Delta_{a_t}|\Theta_h, \mathcal{P}_{a_t}, a_t, h)f(\Theta_h|\mathcal{P}_{a_{t-1}}) \quad (3.32)$$

starting from a given prior pdf of parameters $f(\Theta_h|h) \equiv f(\Theta_h|h, \mathcal{P}_{a_1})$.

The equivalent batch version is, $h \in h^*$,

$$\begin{aligned} f(h|\mathcal{P}_{a_{t+1}}) &\propto \frac{\mathcal{I}_h(\mathcal{P}_{a_{t+1}})}{\mathcal{I}_h(\mathcal{P}_{a_1})} f(h|\mathcal{P}_{a_1}) \\ \mathcal{I}_h(\mathcal{P}_a) &\equiv \int \mathcal{L}(\Theta_h, \mathcal{P}_a) f(\Theta_h|h) d\Theta_h \\ \mathcal{L}(\Theta_h, \mathcal{P}_{a_{t+1}}) &= \prod_{\tau=1}^t f(\Delta_{a_\tau}|\Theta_h, \mathcal{P}_{a_\tau}, a_\tau, h), \quad \mathcal{L}(\Theta_h, \mathcal{P}_{a_1}) \equiv 1. \end{aligned} \quad (3.33)$$

Proof: It uses basic algebra with pdfs, Proposition 2.3.2.2, respecting natural conditions of acting adopted for all involved decisions. \diamond

Problem 3.3.3.4 (How to treat select the list of hypotheses?) *Mechanical ways of generating list of hypotheses make \hat{h} extremely large and consequently the their testing infeasible. GUHA methodology represents a systematic way how to proceed. Nothing, however, has been elaborated which would fit the methodology presented here.*

Problem 3.3.3.5 (How to treat extending list of hypotheses?) *Hypotheses are usually created gradually. It opens a question, how to extend the existing set of hypothesis and how to exploit former data so that the new hypothesis is compared in a fair way. A lot of partial steps have been done in this respect but a systematic design and analysis are missing.*

3.3.4 One-step-ahead prediction

One-step-ahead prediction, an extrapolation of experience to the potentially observable data that are part of the ignorance set, is a very basic task that fits in our framework as follows.

- $\mathcal{Q} \equiv (\mathcal{P}_{\hat{\Delta}}, \hat{\Delta}, \Delta) \equiv$
(data at disposal, prediction, potentially observable data).
- Admissible rules are $\mathcal{R} : \mathcal{P}_{\hat{\Delta}}^* \rightarrow \hat{\Delta}^*$.
- Loss function \mathcal{Z} measures a distance of $\hat{\Delta}$ and Δ and may depend on $\mathcal{P}_{\hat{\Delta}}$.

The outer system model (2.20) needed is $f(\Delta|\mathcal{P}_{\hat{\Delta}})$ as the natural conditions of decision making (2.36) imply that the decision $\hat{\Delta}$ is superfluous in conditioning. This pdf is obtained according to Proposition 2.5.2.1 as the generalised Bayesian prediction. For a given $\mathcal{P}_{\hat{\Delta}}$, the optimal point prediction is, Proposition 2.3.3.1,

$$\hat{\Delta}(\mathcal{P}_{\hat{\Delta}}) \in \text{Arg} \min_{\hat{\Delta} \in \hat{\Delta}^*} \int \mathcal{Z}(\mathcal{P}_{\hat{\Delta}}, \hat{\Delta}, \Delta) f(\Delta|\mathcal{P}_{\hat{\Delta}}) d\Delta. \quad (3.34)$$

Remark(s) 3.3.4.1

1. The prediction experience may contain results of other decisions, like system inputs. In this case $\mathcal{P}_{\hat{\Delta}} \equiv (\mathcal{P}_u, u) \equiv D$, inputs are integral part of experience and they influence both value of the innovation and its prediction.
2. For the popular quadratic loss $\mathcal{Z}(D, \hat{\Delta}, \Delta) = \|\Delta - \hat{\Delta}\|_{Q(D)}^2$ with quadratic norm $\|\cdot\|_{Q(D)}$ weighted by a positive definite kernel $Q(\cdot)$, see Remarks 3.3.1.1 and the relation (3.26), the conditional expectation is the optimal prediction for any weight $Q(\cdot)$,

$$\hat{\Delta}(D) = \mathcal{E}[\Delta|D]. \quad (3.35)$$

3. The discussed prediction should be called point prediction. As in the estimation, set estimates (credibility intervals) might be searched for. Their discussion is identical with that presented in Section 3.3.2.

3.3.5 One-step-ahead control

One-step-ahead control tries to influence the response of the controlled system in the nearest future by choosing an appropriate input. This problem is formulated within our framework as follows.

- $\mathcal{Q} \equiv [\mathcal{P}_u, u, (y, x)] \equiv$
[data at disposal, system input, (system output, system state)].
- Admissible rules are of the form $\mathcal{R} : \mathcal{P}_u^* \rightarrow u^*$.
- Loss function \mathcal{Z} measures distance of (y, x, u) to desirable known reference values (y_r, x_r, u_r) and may depend on \mathcal{P}_u .

The outer system model (2.20) needed is $f(y, x|\mathcal{P}_u, u) = f(x|\mathcal{P}_u, u, y)f(y|\mathcal{P}_u, u)$. The factors creating it are obtained by applying Proposition 2.5.1.3 if we

- set $x \equiv \Theta_t$,
- take the filtration result after data updating as the first factor,
- use the output predictor $f(y|\mathcal{P}_u, u)$ as the second factor.

For the given \mathcal{P}_u , the optimal inputs are (Proposition 2.3.3.1)

$$u(\mathcal{P}_u) \in \text{Arg min}_{u \in u^*} \int \int \mathcal{Z}(\mathcal{P}_u, y, x, u, y_r, x_r, u_r) f(x|\mathcal{P}_u, u, y) f(y|\mathcal{P}_u, u) dx dy. \quad (3.36)$$

Remark(s) 3.3.5.1

1. The input u influences directly the outer model of the system. This property makes the main difference of the control task from estimation.
2. This case is a special version of the multi-step control design, see Section 3.4.4, where additional details are presented.

3. Often, just output of the system is controlled. Then the output predictor is needed only.
4. Output and state reference values are often called set points.
5. Quadratic distance is a popular option used. Then, the conditional expected loss optimized depends on 1st and 2nd conditional moments of the outer model. Often, the conditional variance does not depend on the chosen input. Then, a quadratic form in the conditional expectation $\mathcal{E}\{[y, x]|\mathcal{P}_u, u\}$ is just needed for determining the optimal input.
6. Often, multi-step-ahead control design is approximated by using an extreme version of receding-horizon strategy, see Section 4.4.1. The substantial loss of quality up to the closed-loop instability may be encountered in this case [9]. This extreme should be avoided as much as possible.
7. Fully probabilistic version, see Proposition 2.4.2.4, of this problem may be easily formulated and solved. It is the simple application of Proposition 2.4.2.4 and as such is omitted here.

3.4 Dynamic tasks

Dynamic decision making discussed here form the core of this work. This justified a bit more detailed discussion.

3.4.1 Sequential estimation

Mostly, we consider decision tasks with a fixed horizon $\hat{t} < \infty$. Sequential estimation is an important exception.

If technically possible, it is often advantageous to decide whether to perform a new measurement before making the final decision. This task makes sense if we respect the price paid in connection with the data acquisition. The optimal decision strategies, called *sequential*, then lead in average to a shorter observation process. It makes the overall decision making cheaper in comparison with strategies that rely on a pre-defined sufficiently long decision horizon. Sequential estimation serves well for describing the idea. Other variants, like sequential testing of hypothesis might be formulated and solved similarly.

Sequential point estimation can be cast in our framework as follows.

- $\mathcal{Q} \equiv [\mathcal{P}_{a_t}, a_t \equiv (\hat{\Theta}_t, s_t), \mathcal{F}_{a_t} \equiv (\Theta, \Delta_{a_t})]$
[data at disposal, (estimate, stopping flag), (unknown parameter, innovations)].
- Admissible strategies consist of rules $\mathcal{R}_t : \mathcal{P}_{a_t}^* \rightarrow (\hat{\Theta}_t^*, s_t^*)$, $\Theta^* \subset \hat{\Theta}^*$, $s_t^* \equiv \{\text{stop measuring and estimate } \Theta, \text{ wait for a new measurement}\} \equiv \{0, 1\}$,
- Loss function

$$\mathcal{Z} = \begin{cases} \sum_{\tau \leq t} c(\mathcal{P}_{a_\tau}) + z(\mathcal{P}_{a_t}, \Theta, \hat{\Theta}_t) & \text{if } s_t = 0 \text{ and } s_\tau = 1, \forall \tau < t \\ \sum_{\tau \leq t} c(\mathcal{P}_{a_\tau}) & \text{if } s_\tau = 1 \forall \tau \leq t \end{cases} \quad (3.37)$$

where $z(\mathcal{P}_{a_t}, \Theta, \hat{\Theta}_t)$ measures a distance of Θ and its estimate $\hat{\Theta}$. $c(\mathcal{P}_{a_\tau})$ denotes a positive price of τ th observation.

Proposition 3.4.1.1 (Sequential estimation) *Let us consider the sequential estimation and assume that there is an admissible strategy for which the expected loss is finite. The following inequalities express the sufficient condition for an index t to be the time moment at which observation should be stopped*

$$\mathcal{E} \left[\left(z(\mathcal{P}_{a_t}, \Theta, \hat{\Theta}_t) - z(\mathcal{P}_{a_{t+k}}, \Theta, \hat{\Theta}_{t+k}) - \sum_{\tau>t}^{t+k} c(\mathcal{P}_{a_\tau}) \right) | \mathcal{P}_{a_t} \right] \leq 0, \quad \forall k = 1, 2, \dots \quad (3.38)$$

In (3.38), $\hat{\Theta}_{t+k}$, $k = 0, 1, 2, \dots$ denote parameter estimates based on $\mathcal{P}_{a_{t+k}}$ minimizing $\mathcal{E}[z(\mathcal{P}_{a_{t+k}}, \Theta, \hat{\Theta})]$.

Proof: Let (3.38) be fulfilled. Then, combining the form of the loss function (3.37), the fact that the optimal stopping time has to be determined using its experience and finiteness of the loss for the optimal solution we get, $\forall k = 1, 2, \dots$,

$$\mathcal{E} \left[\left(z(\mathcal{P}_{a_t}, \Theta, \hat{\Theta}_t) + \sum_{\tau=1}^t c(\mathcal{P}_{a_\tau}) \right) | \mathcal{P}_{a_t} \right] \leq \mathcal{E} \left[\left(z(\mathcal{P}_{a_{t+k}}, \Theta, \hat{\Theta}_{t+k}) + \sum_{\tau=1}^{t+k} c(\mathcal{P}_{a_\tau}) \right) | \mathcal{P}_{a_t} \right].$$

Using isotonicity of the expectation (taken over \mathcal{P}_{a_t}), see Proposition 2.3.2.5, we find that the chosen decision cannot be improved by any estimate that uses more measurements than the inspected one.

◇

The outer system model (2.20) needed for this decision making coincides with the generalised Bayesian estimate (posterior pdf) given in Proposition 2.5.2.1.

Remark(s) 3.4.1.2

1. *The ability to evaluate $\mathcal{E} [z(\mathcal{P}_{a_{t+k}}, \Theta, \hat{\Theta}_{t+k}) | \mathcal{P}_{a_t}]$ is decisive for a practical solvability of the problem.*
2. *Stopping rules used for speeding up extensive simulations [48] based on a simple sequential estimation serve as an example of their, still underestimated, usefulness.*
3. *The dependence of the observation price can be effectively exploited when the sequential estimation is performed in an inner loop of some optimization process: the closer we are to the optimum the lower this weight is.*

3.4.2 Multi-step-ahead prediction

Multi-step-ahead prediction, an extrapolation of experience to data in a more distant part of the ignorance set, is an extension of one-step-ahead prediction, Section 3.3.4, and fits in our framework as follows.

- $\mathcal{Q} \equiv (\mathcal{P}_{\hat{\Delta}_{t+j|t}}, \hat{\Delta}_{t+j|t}, \Delta_{t+j}) \equiv$
(data at disposal at time t , prediction of innovations at time $t + j$ based on data at disposal, innovation potentially observable at time $t + j$, $j > 1$).
- Admissible strategies are form by rules $\mathcal{R}_t : \mathcal{P}_{\hat{\Delta}_{t+j|t}}^* \rightarrow \hat{\Delta}_{t+j|t}^*$.
- Loss function \mathcal{Z} measures $\mathcal{P}_{\hat{\Delta}_{t+j|t}}$ -dependent distance of $\hat{\Delta}_{t+j|t}$ and Δ_{t+j} .

The outer system model (2.20) needed is $f(\Delta_{t+j}|\mathcal{P}_{\hat{\Delta}_{t+j|t}})$ and the optimal point prediction is (Proposition 2.3.3.1)

$$\hat{\Delta}_{t+j|t} \in \text{Arg} \min_{\hat{\Delta} \in \hat{\Delta}^*} \int \mathcal{Z}(\mathcal{P}_{\hat{\Delta}_{t+j|t}}, \hat{\Delta}, \Delta_{t+j}|j) f(\Delta_{t+j}|\mathcal{P}_{\hat{\Delta}_{t+j|t}}) d\Delta_{t+j}.$$

Let us illustrate the construction of the needed outer model in a special case of the controlled system with $\Delta_t = y_t =$ system output. It is controlled by a randomized controller described by its outer model $f(u_t|\mathcal{P}_{u_t})$, Agreement 2.3.3.2. The outer model of the system $f(y_t|\mathcal{P}_{u_t}, u_t)$ is either given or obtained as the predictive pdf by filtering, Proposition 2.5.1.3 or through parameter estimation, Proposition 2.5.2.1. The experience of the multi-step predictor is assumed to coincide with \mathcal{P}_{u_t}, u_t .

Using the calculus with pdfs, Proposition 2.3.2.2, the needed outer model can be expressed with the help of the available elements as follows

$$\begin{aligned} f(y_{t+j}|u_t, \mathcal{P}_{u_t}) &= \\ &= \int f(y(t-t+j), u(t+1_t+j)|\mathcal{P}_{u_t}, u_t) d(y(t-t+j-1), u(t+1_t+j)) = \\ &= \int \prod_{\tau=t}^{t+j} f(y_\tau|\mathcal{P}_{u_\tau}, u_\tau) f(u_\tau|\mathcal{P}_{u_\tau}) d(y(t-t+j-1), u(t+1_t+j)). \end{aligned}$$

Remark(s) 3.4.2.1

1. *The need to know the future control strategy makes the main difference of this task from one-step-ahead prediction where just knowledge of inputs is sufficient.*
2. *As a rule, the computed marginal pdf $f(y_{t+j}|u_t, \mathcal{P}_{u_t})$, $j > 1$ is much flatter than the one-step ahead predictor. Formally, it results from integration (averaging) over intermediate predicted values. It reflects the fact that it is much harder (less reliable) to make a long term prediction. The situation is the more pronounced the longer the prediction horizon j is.*
3. *Notice that integrations over the intermediate quantities is done also over their values in conditioning. This makes multi-step prediction highly non-linear task.*
4. *Sometimes, the parameterized model can be constructed directly with the gap j , i.e. in the form $f(\Delta_{t+j}|\Theta, \mathcal{P}_{\hat{\Delta}_{t+j|t}})$. Then, there is no formal difference to the one-step-ahead prediction. The prediction quality is of course worse. If, however, there is a gap in system-output sequence and no gap in system-input sequence the prediction of future inputs is necessary.*

Predictors of this type are used in connection with so called MUSMAR controllers [49].

5. *The point prediction is discussed here. Of course, the set estimation may be also formulated and solved using our general tools.*

3.4.3 Filtering derivatives

Filtering, described in Proposition 2.5.1.3, provides filtration and one step-ahead prediction of an unknown internal variable Θ_t and can be extended to multi-step prediction in the same manner as the innovation prediction discussed in Section 3.4.2. The fact that we never observe directly time varying Θ_t calls for a novel task called *smoothing*.

A smoother estimates the internal quantities Θ_{t-j} , $j > 1$ using also the measured data reflecting newer variables $\Theta((t-j+1)-t)$. Its construction is cast in our framework as follows.

- $\mathcal{Q} \equiv (\mathcal{P}_{\hat{\Theta}_{t-j|t}}, \hat{\Theta}_{t-j|t}, \Theta_{t-j}) \equiv$
(data at disposal at time t , smoothed estimate of Θ_{t-j} based on data at disposal, unknown internal variable at time $t-j$, $j \geq 1$).
- Admissible rules are of the form $\mathcal{R} : \mathcal{P}_{\hat{\Theta}_{t-j|t}}^* \rightarrow \hat{\Theta}_{t-j|t}^*$.
- Loss \mathcal{Z} measures distance of $\hat{\Theta}_{t-j|t}$ and Θ_{t-j} and may depend on $\mathcal{P}_{\hat{\Theta}_{t-j|t}}$.

Assuming controlled system and $\mathcal{P}_{\hat{\Theta}_{t-j|t}} \equiv (u_t, \mathcal{P}_{u_t})$, basic decision making lemma, Proposition 2.3.3.1, reads

$$\hat{\Theta}_{t-j|t} \in \text{Arg} \min_{\hat{\Theta} \in \hat{\Theta}_{t-j}^*} \int \mathcal{Z}(u_t, \mathcal{P}_{u_t}, \hat{\Theta}_{t-j}, \Theta_{t-j}) f(\Theta_{t-j} | \mathcal{P}_{u_t}, u_t) d\Theta_{t-j}$$

Let us assume that we have at disposal both observation $f(y_t | \Theta_t, \mathcal{P}_{u_t}, u_t)$ and time evolution $f(\Theta_t | \Theta_{t-1}, u_t, \mathcal{P}_{u_t})$, $t \in t^*$ models used in Proposition 2.5.1.3. The applied control strategy meets natural conditions of decision making (2.36). Using the calculus with pdfs, Proposition 2.3.2.2, and (2.36), the model needed in the above optimization can be expressed with the help of the available elements as follows

$$\begin{aligned} f(\Theta_{t-j} | \mathcal{P}_{u_t}, u_t) &\propto f(\Theta_{t-j}, y(t-j-t), u(t-j-t) | \mathcal{P}_{u_{t-j}}) = \\ &= \int \prod_{\tau=t-j}^t f(y_\tau | \Theta_\tau, \mathcal{P}_{u_\tau}, u_\tau) f(\Theta_\tau | \Theta_{\tau-1}, \mathcal{P}_{u_\tau}, u_\tau) f(\Theta_{t-j} | \mathcal{P}_{u_{t-j}}), d\Theta(t-j+1-t) \end{aligned}$$

The integrand contains the available models with available data substituted. The factor $f(\Theta_{t-j} | \mathcal{P}_{u_{t-j}})$ results from filtering described by Proposition 2.5.1.3. Smoothing is simply derived from filtering.

3.4.4 Multi-step-ahead control

Multi-step-ahead control, and extension of one-stage-ahead control of Section 3.3.5, tries to push quantities characterizing the controlled system, namely (system output, system input, system state) = (y, u, x) , to the desired reference values (y_r, u_r, x_r) . Their proximity should be enforced on a whole time-interval up to the horizon \hat{t} by choosing an appropriate sequence of inputs $u(\hat{t})$.

We consider here the most general combination of tracking and regulation to uncertain reference values. The specialization to the case of (partially) known reference values is straightforward. This problem is formulated in our framework as follows.

- $\mathcal{Q} \equiv [\mathcal{P}_{u_t}, u_t, \mathcal{F}_{u_t}] \equiv$
 [(observed data $\equiv y(t-1), u(t-1), y_r(t-1), u_r(t), x_r(t-1)$), system input u_t , (future system outputs $y(t-\dot{t})$, states $x(\dot{t})$ and references $y_r(t-\dot{t}), u_r(t+1-\dot{t}), x_r(t-\dot{t})$)].

Time t is increasing within the set t^* .

- Admissible strategy consists of a sequence *control laws* (decision rules) of the form $\{\mathcal{R}_t : \mathcal{P}_{u_t}^* \rightarrow u_t^*\}_{t \in t^*}$.
- Loss \mathcal{Z} measures distance of \mathcal{Q} to reference values $y_r(\dot{t}), u_r(\dot{t}), x_r(\dot{t})$.

The optimal control strategy can be designed using directly dynamic programming, see the general version in Proposition 2.4.1.1, the version with additive loss in Proposition 2.4.1.5 or the fully probabilistic version in Proposition 2.4.2.4.

The needed outer model is $f(y_t, x_t, y_{r;t}, x_{r;t} | \mathcal{P}_{u_t}, u_t, u_{r;t})$. It is obtained via filtering with unknown internal variables $\Theta_t = (x_t, y_{r;t}, x_{r;t})$ using Proposition 2.5.1.3.

Remark(s) 3.4.4.1

1. *The optimal controller consists of the combination of filtering and controller acting on its results.*
2. *Control of the system with unknown time invariant parameters Θ , the central topic of main stream adaptive control [3], is a special case of the described design with the trivial time evolution $\Theta_t = \Theta_{t-1}$. The parameter estimation, see Proposition 2.5.1.3, provides the necessary outer system model in this case.*
3. *It is fair to say that the optimal multi-step-ahead control has the widest gap between the optimal design and practically optimal design. It is rarely analytically or numerically feasible. For this reason, a lot of heuristic approximation techniques have been developed. Some of them are discussed in Chapter ??.*
4. *The time evolution model $f(x_t, y_{r;t}, x_{r;t} | x_{t-1}, y_{r;t-1}, x_{r;t-1}, \mathcal{P}_{u_t}, u_t, u_{r;t})$ has to describe the evolution of both system state and uncertain reference values.*
5. *The typical control design optimizing closed-loop behaviour with respect to randomly occurring step changes can be obtained by modelling the output and/or state references as random walks with time-varying dispersion [50].*
6. *The described general cases covers the situation when we try to follow evolution of another uncertain object. Rescue/military interpretations of this case are straightforward. The key message is that the dynamics of the uncertain target has to be modelled for finding the optimal strategy.*

The situation is simplified whenever target values are (partially) known. Then the corresponding outer models reduce formally to Dirac delta functions on a given known support.

Chapter 4

Evaluation techniques

Here, we review basic evaluation techniques and tools that create *bridge between optimal and practically optimal design*. All of them try to cope with the complexity restrictions. The complexity of the optimal design stems from the fact that the optimal “plan” of dynamic decisions is essentially searched for by comparing all possible behaviours resulting from the interaction of the system and judged strategy. The design complexity is significantly influenced by richness of the inspected space. Its reduction is behind the majority of available approximation schemes including those discussed here.

4.1 Adaptive systems

The ideal solution of the decision making under uncertainty is described by the combination of Bayesian filtering, see Proposition 2.5.1.3, and dynamic programming, see Propositions 2.4.1.5, 2.4.2.4. The functional equations describing them are mostly computationally infeasible. Consequently, their use cannot live without approximations. Even the simpler fully probabilistic design operates on infinite-dimensional objects. The involved multivariate functions should be represented in computer, i.e. in the device that can operate on a high but finite amount of values. Thus, a sort of approximation is needed. Global approximation of functions of many variables we are dealing with is known to be computationally hard. The application of the strategies resulting from the design requires knowledge of the discussed solutions only for the recorded experience. Thus, it is sufficient to know them *locally around the actual experience*. It can often be approximately achieved. Such *local approximators* are known as *adaptive systems* [51]. They naturally exploit majority approximation techniques described below.

Remark(s) 4.1.0.2

1. Note that there is no formal definition of adaptive systems. Their operational description is, for instance, in [3].
2. We found the coined understanding of adaptive systems as local approximators very useful. It helps us to have a unified view on existing practical strategies and opens a way for designing novel ones [51]. Moreover, it shows that the adaptive systems will be inevitably used in future due to the theoretically provable need for local approximations.

4.2 Suboptimal estimation

Parameter estimation, Proposition 2.5.2.1, is a special case of filtration, Proposition 2.5.1.3, with a “trivial” time-evolution model $f(\Theta_{t+1}|\Theta_t, \mathcal{P}_{a_{t+1}}, a_{t+1}) = \delta(\Theta_{t+1} - \Theta_t)$. Consequently, majority of techniques suitable for approximate filtering, see Section 4.3, can be used in this case, too. Specific common techniques related to estimation only are outlined here.

4.2.1 Equivalence approach

Under natural conditions of decision making (2.36), the generalised Bayesian estimation (Proposition 2.5.2.1) updates the posterior pdfs according to the Bayes rule (2.43)

$$f(\Theta|\mathcal{P}_{a_{t+1}}) = \frac{f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)f(\Theta|\mathcal{P}_{a_t})}{f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t)}, \quad t \in t^*. \quad (4.1)$$

Complexity of expressing of this pdf increases quickly with an increasing amount of data, with increasing t . Exponential family (2.53) is essentially the only exception from this rule. This section tries to cope with *recursive estimation out of exponential family*.

Always limited capabilities of a computer that treats non-exponential parameterised models call for a reduced representation of propagated posterior pdfs. It is a peculiar task as the posterior pdfs concentrate quickly on a very narrow support at unknown position in Θ^* , see Proposition ???. Thus, a representation on a sufficiently fine grid that does not miss the final position becomes soon computationally prohibitive. The way out has been elaborated in a sequence of papers by Dr. Kulhavý and summarised in [24]. Here, we just outline the essence of this *equivalence approach*.

Proposition 4.2.1.1 (Equivalence-preserving mapping) *Let g_t be a finite-dimensional vector representing the posterior pdf $f(\Theta|\mathcal{P}_{a_t})$. In other words, g_t is a finite-dimensional image generated by a mapping $G_t : (f(\Theta|\mathcal{P}_{a_t}))^* \rightarrow g_t^*$. Let the parameterised models be positive on a common, time, data and parameter invariant support.*

Then, the values of g_t can be exactly recursively updated using only its previous value and the current parameterised model $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)$ iff G_t is a time-invariant linear mapping $G \equiv G_t \quad t \in t^$ acting on logarithms of the pdfs involved. The logarithmic pdfs are treated as functions of Θ . G_t has to map Θ -independent elements to zero.*

Proof: To demonstrate necessity is rather hard and the interested reader is referred to [52, 53]. To show that the conditions on $G_t \equiv G, t \in t^*$, are sufficient is simple and instructive. They become obvious if we apply G to the logarithmic version of the Bayes rule (4.1) and use both its time-invariance and linearity. The normalising term $\ln(f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t))$ is independent of Θ and as such mapped to zero. The recursion for g_t is then simply

$$g_t = G[\ln(f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t))] + g_{t-1}, \quad \text{with } g_0 = G(f(\Theta)) \equiv G(\text{prior pdf}). \quad (4.2)$$

◇

Note that (4.2) becomes the true recursion if we need not store whole past observed data for evaluating $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t)$. Thus, similarly as in the case of exponential family, see Agreement 2.5.4.1, we adopt the mild assumption that

$$f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) \equiv M(\Theta, \Psi_t). \quad (4.3)$$

The data $\Delta_{a_t}, a_t, \mathcal{P}_{a_t}$ are compressed into a finite dimensional data vector Ψ_t that can be updated recursively $\Psi_{t-1}, a_t, \Delta_{a_t} \rightarrow \Psi_t$.

In the subsequent discussion, we need:

Agreement 4.2.1.2 (Empirical pdf; Riezs representation) *Let parameterised model have the form (4.3). Then,*

$$f_t(\Psi) \equiv \frac{1}{t} \sum_{\tau=1}^t \delta(\Psi - \Psi_\tau), \quad \Psi \in \Psi^* \equiv \bigcup_{t \in t^*} \Psi_t^* \quad (4.4)$$

is the (formal) empirical pdf of Ψ . The used Dirac delta function $\delta(\cdot)$ is defined in (??).

We assume that the same representation exists for the linear mapping G introduced in Proposition ??, i.e. there is a possibly generalised vector function $G(\Theta)$ such that

$$G(C) \equiv \int_{\Theta^*} C(\Theta) G(\Theta) d\Theta \quad (4.5)$$

for any considered function $C(\Theta)$, $\Theta \in \Theta^*$.

Assuming (4.3), using the (formal) empirical pdf (4.4) and Riezs representation of G (4.5), we see that

$$g_t = t \int_{\Psi^*} \left[\int_{\Theta^*} \ln[M(\Theta, \Psi)] G(\Theta) d\Theta \right] f_t(\Psi) d\Psi \quad (4.6)$$

for the vector function $G(\Theta)$ representing functionals chosen. The integral in brackets $[\]$ defines the vector function

$$h(\Psi) \equiv \int_{\Theta^*} \ln[M(\Theta, \Psi)] S(\Theta) d\Theta. \quad (4.7)$$

With it, we get the equivalent form of (4.6)

$$g_t = t \int_{\Psi^*} h(\Psi) f_t(\Psi) d\Psi + g_0. \quad (4.8)$$

Recall, that left hand side of (4.8) is known as it can be update recursively according to (4.2) which has the equivalent ‘‘Riezs’’ form

$$g_t \underset{(4.2)}{\equiv} \int_{\Theta^*} \ln[M(\Theta, \Psi_t)] G(\Theta) d\Theta + g_{t-1} \underset{(4.7)}{\equiv} h(\Psi_t) + g_{t-1}, \quad g_0 = G(f(\Theta)). \quad (4.9)$$

Also, for the chosen parameterised model (4.3) and functions $G(\Theta)$ representing the admissible projections to g_t^* , the vector function $h(\Psi)$ is known as well as its values in measured data vectors Ψ_t . The empirical pdf $f_t(\Psi)$ (4.4) is the only incompletely known object in (4.8) as we are not able (or willing) to store all measured Ψ_τ , $\tau \leq t$.

The posterior pdf, we are interested in, can be expressed in terms of empirical pdf as follows, cf. similar transformation in connection with Proposition ??,

$$f(\Theta | \mathcal{P}_{a_{t+1}}) \equiv f(\Theta | f_t) \propto f(\Theta) \exp \left[t \int_{\Psi^*} \ln[M(\Theta, \Psi)] f_t(\Psi) d\Psi \right]. \quad (4.10)$$

The empirical pdf is unknown to us, thus this posterior pdf is unknown, too. The selection of a suitable approximating pdf $\hat{f}(\Theta | g_t)$ acting on Θ^* fits to our general decision making framework:

- $\mathcal{Q} \equiv (\mathcal{P}_a, a, \mathcal{F}_a) \equiv (g_t, \hat{f}(\Theta|g_t), f(\Theta|f_t(\cdot))) \equiv$
(stored statistic, estimate of the posterior pdf, the posterior pdf determined by the unknown empirical pdf $f_t(\Psi)$, $\Psi \in \Psi^*$),
- Admissible decision rule are of the form

$$\mathcal{R} : g_t^* \rightarrow \left\{ \hat{f}(\Theta|g_t) \geq 0, \int \hat{f}(\Theta|g_t) d\Theta = 1, \Theta \in \Theta^* \right\}$$

- Loss function is Kulback-Leibler distance (2.26) $\mathcal{D}(\hat{f}(\Theta|g_t)||f(\Theta|f_t))$

Formally, the optimal point estimate of the posterior pdf is, Proposition 2.3.3.1,

$$\begin{aligned} \hat{f}(\Theta|g_t) &\in \text{Arg min}_{\hat{f} \in \hat{f}^*} \mathcal{E} \left[\mathcal{D}(\hat{f}(\Theta|g_t)||f(\Theta|f_t)) | g_t \right] = \\ &= \text{Arg min}_{\hat{f} \in \hat{f}^*} \left[\mathcal{D}(\hat{f}(\Theta|g_t)||f(\Theta)) - t \int_{\Psi^*} \hat{f}(\Theta|g_t) \ln[M(\Theta, \Psi)] \mathcal{E}[f_t(\Psi)|g_t] d\Psi \right]. \end{aligned} \quad (4.11)$$

The conditional expectation $\mathcal{E}[\cdot|g_t]$ is taken with respect to the uncertain empirical pdf $f_t(\Psi)$, $\Psi \in \Psi^*$.

The minimiser can be found explicitly as

$$\hat{f}(\Theta|g_t) \propto f(\Theta) \exp \left\{ t \int_{\Psi^*} \ln[M(\Theta, \Psi)] \mathcal{E}[f_t(\Psi)|g_t] d\Psi \right\}. \quad (4.12)$$

It depends only on the conditional expectation $\mathcal{E}[f_t(\Psi)|g_t]$. The empirical pdf fulfills the identity (4.8) that is linear in it. Consequently, its conditional expectation $\mathcal{E}[f_t(\Psi)|g_t]$ has to, see Proposition 2.3.2.5, fulfill it, too:

$$g_t = \int_{\Psi^*} h(\Psi) \mathcal{E}[f_t(\Psi)|g_t] d\Psi + g_0. \quad (4.13)$$

For the given g_t , g_0 , the solution (4.12) can be used if we specify the expectation

$$E(\Psi) \equiv \mathcal{E}[f_t(\Psi)|g_t], \Psi \in \Psi^*. \quad (4.14)$$

For it, let us accept

Agreement 4.2.1.3 (Linear span over h -entries; Orthogonality) *The set of real functions defined on Ψ^* and spanned linearly over entries of the vector function $h(\Psi)$, see (4.7)*

$$e^* \equiv \left\{ \sum_{i \in i^* \equiv \{1, \dots, \hat{h}\}} \alpha_i h_i(\Psi) \text{ with arbitrary real weights } \alpha_i \right\} \quad (4.15)$$

is embedded into a linear space E^* with the scalar product $\langle \cdot, \cdot \rangle$. For $A, B \in E^*$ it is defined $\langle A, B \rangle \equiv \int_{\Psi^*} A(\Psi) B(\Psi) d\Psi$.

The orthogonal complement $e^{*\perp}$ of e^* to E^* is the set

$$e^{*\perp} \equiv \{A \in E^* \setminus 0 : \langle A, B \rangle = 0 \forall B \in e^*\}. \quad (4.16)$$

The restriction (4.12) on the constructed expectation $E(\Psi)$ (4.14) can be written in the equivalent form

$$g_{i;t} - g_{i;0} = \langle h_i, E \rangle, \quad \forall i \in i^* \equiv \{1, \dots, \overset{\circ}{h}\}. \quad (4.17)$$

This form and definition (4.16) imply that if some $E(\cdot) \in E^*$ fulfills (4.17) then also $E(\cdot) + B(\cdot)$ does whenever $B(\cdot) \in e^{*\perp}$. All such options are equivalent. At the same time, we have exploited all relevant information sources: the prior pdf $f(\Theta)$, the parameterised model $M(\Theta, \Psi)$ and data compressed into the recursively feasible statistics g_t . We have no reason to prefer any $B(\cdot) \in e^{*\perp}$. Consequently, we consider that $E(\cdot)$ (refexpeE)

$$E(\Psi) \in e^* \Leftrightarrow \mathcal{E}[f_t(\Psi)|g_t] = h'(\Psi)\alpha_t, \quad \text{for an } \overset{\circ}{h}\text{-vector of weights } \alpha_t(g_t). \quad (4.18)$$

This assumption and the condition (4.17) determines the weights as the solution of the set of linear algebraic equations

$$g_t - g_0 = H\alpha_t, \quad H_{ij} \equiv \langle h_i, h_j \rangle, \quad i, j \in i^*. \quad (4.19)$$

The constant square matrix H is positive semidefinite. Its regularity can simply be guaranteed by a proper choice of $h(\cdot)$. Then, the weights α and consequently of $E(\cdot)$ are uniquely determined. Thus, we can summarise the overall algorithm:

Algorithm 4.2.1.4 (Recursive estimation based on equivalence approach) Off-line phase

1. Select the parameterised model $f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) \equiv M(\Theta, \Psi_t)$.
2. Select the prior pdf $f(\Theta)$.
3. Select the (generalised) vector function $G(\Theta)$ such that $\int G(\Theta) d\Theta = 0$.
4. Prepare evaluation of the function

$$h(\Psi) \equiv \int \ln(M(\Theta, \Psi))G(\Theta) d\Theta. \quad (4.20)$$

5. Compute the positive definite matrix $A \equiv [\int h(\Psi)h'(\Psi), d\Psi]^{-1}$.
6. Prepare evaluation of the function

$$H(\Theta) \equiv \int \ln(M(\Theta, \Psi))h(\Psi) d\Psi. \quad (4.21)$$

7. Set $t = 0$ and α_0 .

On-line use phase for $t \in t^*$

1. Measure data vector Ψ_t .
2. Evaluate $h(\Psi_t) \equiv \int \ln(M(\Theta, \Psi_t))G(\Theta) d\Theta$.
3. Update weights $\alpha_t = Ah_t + \alpha_{t-1}$.
4. Exploit the approximation of the posterior pdf

$$\hat{f}(\Theta|g_t) \propto f(\Theta) \exp [H'(\Theta)\alpha_t]. \quad (4.22)$$

Remark(s) 4.2.1.5

1. Zero initial weights α_0 reflect the fact that they are always determined by the increment $g_t - g_0$, see (4.19).
2. The term “equivalence” approach stresses the fact that the set of posterior pdfs is reduced to equivalence classes: pdfs with the same representation g cannot be distinguished.
3. The chosen form of the expectation (4.18) determining the final solution is definitely not unique. It has, however, the smallest norm among possible options and as such it has tendency to give the most flat estimate of the posterior pdf. This seems to be wise as the approximation involved adds uncertainty to the estimation results.
4. The required commutativity of the updating and projecting the posterior pdfs is crucial. The recursion for g_s is exact and the approximation errors caused by the use of $\hat{f}(\Theta|g_t)$ instead of $f(\Theta|\mathcal{P}_{a_{t+1}}) \equiv f(\Theta|f_t(\cdot))$ do not accumulate! Use of a non-commutative projection $G_t : f^*(\Theta|\mathcal{P}_{a_t}) \rightarrow g_t^*$ is always endangered by divergence: the estimation described by the Bayes rule can be viewed as a dynamic system evolving $f(\Theta|\mathcal{P}_{a_{t+1}})$ at stability boundary.
5. The indicated integrations represent computationally the most demanding part of the algorithm. They can be performed in off line mode if their results can be efficiently stored (the resulting functions interpolated).

Problem 4.2.1.6 (How to choose the mapping G ?) The (generalised) functions G represent the key tuning knobs of the approach. Options leading to discretisation of the function and/or its derivatives, or $M(\Theta, \Psi_i)$ on a grid of Ψ_i have been tried with a success but a deeper insight is needed in order to arrive to a cook-book.

Problem 4.2.1.7 (Does exist an equivalent for filtration?) There are indications that answer is negative. If it is true, other approaches relying on usual stabilising effect of time-updating (2.40) have to be used. They, however, should have a smooth transition to the estimation as its special case.

4.2.2 Estimation with forgetting

The case of *slowly varying parameters* is discussed here as an intermediate, widely met, case bridging estimation and filtering. It admits time variations of Θ_t but it assumes that $\Theta_{t+1} \approx \Theta_t$. At the same time, a more precise model of changes is not used.

We summarise an approach called *stabilised forgetting*, [54]. It is based on a flexible problem formulation and a simple application of Proposition 3.1.4.8. It proved to be a powerful tool for solving the considered ill-posed problem.

Let $f(\Theta_{t+1} = \Theta_t = \Theta|\mathcal{P}_{a_{t+1}})$ be pdf that assumes no parameter changes and $f_a(\Theta_{t+1} = \Theta|\mathcal{P}_{a_{t+1}})$ an *alternative pdf* that describes parameters after some changes. Let $\phi \in (0, 1]$ be the probability that the “correct” unknown pdf $f(\Theta) \equiv f(\Theta_{t+1} = \Theta|\mathcal{P}_{a_{t+1}})$ has the best projection equal to the former pdf and $1 - \phi$ the probability that the later one is relevant. Then, the best compromise minimising its expected Kullback-Leibler distance to the unknown best projection is, see Proposition 3.1.4.8,

$$f(\Theta_{t+1} = \Theta|\mathcal{P}_{a_{t+1}}) \propto [f(\Theta_{t+1} = \Theta_t = \Theta|\mathcal{P}_{a_{t+1}})]^\phi [f_a(\Theta_{t+1} = \Theta|\mathcal{P}_{a_{t+1}})]^{1-\phi}. \quad (4.23)$$

Using this formula, we approximate the time updating step (2.40) in filtering without explicitly specifying the model of time evolution (2.35).

Algorithm 4.2.2.1 (Stabilised forgetting) *Select the probability $\phi \in (0, 1]$ and specify prior pdf $f(\Theta_1 = \Theta) \equiv f(\Theta_1 = \Theta | \mathcal{P}_{a_1})$ corresponding to the treated parameterised (observation) model $f(\Delta_{a_t} | \Theta_t, \mathcal{P}_{a_t}, a_t)$.*

Repeat for time $t \in t^$*

1. *Collect the newest data $\mathcal{P}_{a_{t+1}}, a_{t+1}$*
2. *Perform data updating*

$$f(\Theta_t = \Theta | \mathcal{P}_{a_{t+1}}) \propto f(\Delta_{a_t} | \Theta, \mathcal{P}_{a_t}, a_t) f(\Theta_t = \Theta | \mathcal{P}_{a_t}).$$

3. *Select or update the alternative pdf $f_a(\Theta_{t+1} = \Theta | \mathcal{P}_{a_{t+1}})$.*
4. *Approximate time updating (forget)*

$$f(\Theta_{t+1} = \Theta | \mathcal{P}_{a_{t+1}}) \propto [f(\Theta_{t+1} = \Theta_t = \Theta | \mathcal{P}_{a_{t+1}})]^\phi [f_a(\Theta_{t+1} = \Theta | \mathcal{P}_{a_{t+1}})]^{1-\phi}.$$

Remark(s) 4.2.2.2

1. *The time-updated pdf (4.23) is a compromise between the posterior pdf obtained under the hypothesis that Θ_t is time-invariant and an externally supplied alternative f_a . The closer ϕ is to unity the slower changes are expected, i.e. the higher weight the posterior pdf corresponding to the time-invariant case gets.*
2. *Note that the forgetting operation (4.23) preserves the basic property of time-updating: the posterior pdf on parameters propagates without obtaining any new measured information.*
3. *Let us assume that $f_a \propto 1$. Then $f(\Theta_{t+1} = \Theta | \mathcal{P}_{a_{t+1}}) \propto [f(\Theta_{t+1} = \Theta | \mathcal{P}_{a_{t+1}})]^\phi$, i.e. the time-updated pdf is a flattened version of the pdf obtained after data-updating (if $\phi < 1$). It is intuitively appealing as our uncertainty about parameters can hardly decrease without knowing a good time-evolution model (2.35) and with no new information processed.*
4. *It is instructive to inspect the influence of forgetting on old data built in through the observation model. The older data are the higher flattening is applied to the corresponding model. Consequently, the older data influence the estimation results less than new ones. Data are gradually “forgotten”. This explains why the probability ϕ is referred as a forgetting factor,*
5. *The alternative pdf assigned to to Θ^* expresses our belief where the parameters might move within the time interval $[t, t + 1)$ while we have no new observable information.*

Often, the pessimistic uniform alternative pdf ($\propto 1$) has been used. This special case of stabilised forgetting is called exponential forgetting. It allows us to follow relatively fast parameter changes but it forgets the accumulated information with, often to high, exponential rate. For this reason, it is worth to preserve what we feel as a guaranteed information. The prior pdf $f(\Theta_{t+1} = \Theta)$ is a typical, reasonably conservative choice of the alternative pdf $f_a(\Theta_{t+1} = \Theta | \mathcal{P}_{a_t})$.

6. The non-trivial alternative pdf prevents us to forget the “guaranteed” information as it is always incorporated after flattening (exponential forgetting). This stabilises whole learning and reflects very positively in its numerical implementations. Without this, the posterior pdf may become too flat whenever the information brought by new data is not sufficient.

Note that insufficient informativeness of new data is more rule than exception. It is true especially in regulation problems in which the controller tries to make closed-loop as quiet as possible, i.e. it tries to suppress any new information in data.

7. The forgetting factor ϕ can be either taken as tuning knob or estimated. The predictive pdf parameterised by it, however, depends on it in a very complex way so that a partitioned estimation has to be applied when its posterior pdf is estimated on a prespecified grid [55], see Section 4.3.1.

Also, the forgetting factor can be chosen in a pessimistic way as the maximiser of the reached minima, see Proposition 3.1.4.8.

8. Practical importance of this particular case of slowly varying parameters cannot be overstressed: vast majority of adaptive systems, see Section 4.1, rely on a version of forgetting combined with a version of receding horizon strategy, see Section 4.4.1. With it, the estimation and prediction are approximated locally within time domain. They rely on data measured nearby the actual time.

4.2.3 Quasi-Bayes estimation

The quasi-Bayes estimation described here is a special approximation techniques designed for mixtures of models introduced in Section 3.1.3. The standard Bayesian estimation, see Section 2.5.2, is infeasible with this parameterised model

$$f(\Delta_{a_t}|\Theta, \mathcal{P}_{a_t}, a_t) = \sum_{c \in c^*} \alpha_c f_c(\Delta_{a_t}|\Theta_c, \mathcal{P}_{a_t}, a_t), \quad \Theta \in \Theta^* \equiv \left\{ \Theta = (\alpha_c, \Theta_c), \Theta_c \in \Theta_c^*, c \in c^* \equiv \{1, \dots, \dot{c} < \infty\}, \alpha \in \alpha^* \equiv \left\{ \alpha_c \geq 0, \sum_{c \in c^*} \alpha_c = 1 \right\} \right\}$$

as the number of terms in the likelihood function (??) (a product of parameterised models) blows up exponentially. The quasi-Bayes estimation is justified as follows.

Let us have the generalised Bayesian estimate in the form

$$f(\Theta|\mathcal{P}_{a_t}) \propto \prod_{c \in c^*} f(\Theta_c|\mathcal{P}_{a_t}) \alpha_c^{v_{c;t-1}-1} \quad (4.24)$$

with some pdfs $f(\Theta_c|\mathcal{P}_{a_t})$ and scalars statistics $v_{c;t-1} > 0$.

Introducing the random unobservable selector $c_t \in c^*$ of components, we defined extension of the parameterised model to the joint pdf of c_t and innovation (3.7). Using it, the assumed form (4.24), Bayes rule (2.8) and definition of Kronecker symbol $\delta(c, \tilde{c})$, (??), we get

$$\begin{aligned} f(\Theta, c_t|\mathcal{P}_{a_{t+1}}) &\propto f_{c_t}(\Delta_{a_t}|\Theta_{c_t}, \mathcal{P}_{a_t}, a_t) \alpha_{c_t} \prod_{c \in c^*} f(\Theta_c|\mathcal{P}_{a_t}) \alpha_c^{v_{c;t-1}-1} = \\ &= \prod_{c \in c^*} [f_{c_t}(\Delta_{a_t}|\Theta_{c_t}, \mathcal{P}_{a_t}, a_t)]^{\delta(c, c_t)} f(\Theta_c|\mathcal{P}_{a_t}) \alpha_c^{v_{c;t-1} + \delta(c, c_t) - 1}. \end{aligned} \quad (4.25)$$

The result of the data updating for Θ could be gained from this pdf using marginalisation in order to get rid off unobservable c_t , see Proposition 2.3.2.2. It would, however, destroy the product form (4.24) of the posterior pdf. In order to avoid this trap, we simply replace the unknown $\delta(c, c_t)$ by its point estimate. We take the conditional expectation of $\delta(c, c_t)$ as this estimate. It is the optimal point estimate corresponding at least to the quadratic loss, see Remarks 3.3.1.1:

$$\begin{aligned} \delta(c, c_t) &\approx w_{c;t} \equiv \mathcal{E}[c_t = c | \mathcal{P}_{a_{t+1}}] = f(c_t = c | \mathcal{P}_{a_{t+1}}) \propto \\ &\propto (v_{c;t-1} + 1) \int f_c(\Delta_{a_t} | \Theta_c, \mathcal{P}_{a_t}, a_t) f(\Theta_c | \mathcal{P}_{a_t}) d\Theta_{c_t}. \end{aligned} \quad (4.26)$$

Note that the expectation over unknown α is evaluated using formulae related to Dirrchet distribution.

The substitution of a point estimate preserves the product form (4.24) and whole procedure can be prolonged. We summarise the algorithm for exponential family (2.53) for which it is of practical interest. The results of Propositions 2.5.4.3, 3.1.4.4 are used.

Algorithm 4.2.3.1 (Quasi-Bayes estimation in exponential family)

Off-line phase

1. Select the number of components and their structure, i.e. the form of $f_c(\Delta_{a_t} | \Theta_c, \mathcal{P}_{a_t}, a_t)$.
2. Select prior statistics for estimation of parameters of individual components $V_{c,0}, \nu_{c,0} > 0, c \in c^*$.
3. Select prior statistics $v_{c,0} > 0, c \in c^*$ for estimation of the weights α_c of components.

On-line phase, do for $t \in t^*$

1. do for $c \in c^*$

- (a) Collect the newest data Δ_{a_t}, a_t ,
- (b) Construct data vectors $\Psi_{c,t}$,
- (c) Compute weights (4.26)

$$w_{c;t} \propto \frac{\mathcal{I}(V_{c;t-1} + B(\Psi_{c;t}), \nu_{c;t-1} + 1)}{\mathcal{I}(V_{c;t-1}, \nu_{c;t-1})}, \quad (4.27)$$

- (d) Update statistics

$$\begin{aligned} V_{c;t} &= V_{c;t-1} + w_{c;t} B(\Psi_{c;t}), \quad \nu_{c;t} = \nu_{c;t-1} + w_{c;t} \\ v_{c;t} &= v_{c;t-1} + w_{c;t}. \end{aligned} \quad (4.28)$$

2. Evaluate required characteristics of the approximate posterior pdf

$$f(\Theta | \mathcal{P}_{a_{t+1}}) \propto \prod_{c \in c^*} A^{\nu_{c;t}}(\Theta_c) \exp[V'_{c;t} C(\Theta_c)] \alpha^{v_{c;t}-1}. \quad (4.29)$$

Remark(s) 4.2.3.2

1. The used approximation is intuitively appealing. The more we expect that the innovations Δ_{a_t} for given a_t has been generated by some component the higher weight to the increment $B(\Psi_{c;t})$ of the corresponding statistics we assign.
2. It is straightforward and computationally cheap to allow slow parameter changes by employing the stabilised forgetting, see Algorithm 4.2.2.1.
3. Unlike other procedures applied to mixtures, it fits well for on-line adaptive learning of dynamic systems (with non-trivial regressors).
4. All known procedures are sensitive to initiation, [38]. It applies to the quasi-Bayes estimation even in a wider extent. This is inevitable price for its advantages.
5. The components describing multivariate innovations can be always factorised into product of parameterised models describing predicting single entry (single output). Each of these factors may have its structure and may be even of a different type. Thus, we can combine, for instance, regression models with Markov chains. In this way, we get mixture model describing interrelated continuous and discrete data. It is of an extreme practical importance, especially in connection with the universal approximation property of mixtures.

Problem 4.2.3.3 (How to make quasi-Bayes estimation complete?) *The quasi-Bayes algorithm proved to be powerful complement to universal and flexible mixtures. A lot of subproblems still await for their solutions: initiation, structure estimation, multi-step control design etc.*

4.3 Suboptimal filtering

Filtering, see Proposition 2.5.1.3, is the basic technique for generating the outer model of the system needed for (sub)optimal decision making. There is a bunch of complementary general techniques that convert optimal learning into practically applicable suboptimal learning. They are discussed here.

4.3.1 Partitioned approach

This approach [55] suits to cases when the problem formulation, determined by the observation $f(\Delta_{a_t}|\Theta_t, \mathcal{P}_{a_t}, a_t)$ and time evolution $f(\Theta_t|\Theta_{t-1}, \mathcal{P}_{a_t}, a_t)$ models, has the following structure:

- $\Theta_t = (\Theta, x_t) \equiv$
(time-invariant parameters, state of the system)
- filtering of x_t is solvable for any fixed $\Theta \in \Theta^*$
- a finite (low-dimensional) grid $\{\Theta_i \in \Theta^*\}_{i \in i^*}$ “covers” sufficiently the space of possible parameter values Θ^* .

The partitioned approach is described by the following algorithm:

Algorithm 4.3.1.1 (Partitioned filtering)

Off-line phase

Select prior probabilities $f(\Theta_i|\mathcal{P}_{a_1})$ of individual parameter values on the grid $\{\Theta_i\}_{i \in i^*}$ and set time $t = 0$.

On-line phase, do for $t \in t^*$

1. Repeat for $i \in i^*$

(a) Collect the newest data Δ_{a_t}, a_t ,

(b) Perform the conditional data updating

$$\begin{aligned} f(x_t|\mathcal{P}_{a_{t+1}}, \Theta_i) &= \frac{f(\Delta_{a_t}|x_t, \Theta_i, \mathcal{P}_{a_t}, a_t)f(x_t|\mathcal{P}_{a_t}, \Theta_i)}{f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t)} \propto \\ &\propto f(\Delta_{a_t}|x_t, \Theta_i, \mathcal{P}_{a_t}, a_t)f(x_t|\mathcal{P}_{a_t}, \Theta_i). \end{aligned} \quad (4.30)$$

(c) Perform the conditional time updating

$$f(x_t|\mathcal{P}_{a_t}, \Theta_i) = \int_{x_{t-1}^*} f(x_t|x_{t-1}, \Theta_i, \mathcal{P}_{a_t}, a_t)f(x_{t-1}|\mathcal{P}_{a_t}, \Theta_i) dx_{t-1}. \quad (4.31)$$

(d) Evaluate the conditional prediction

$$f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t, \Theta_i) = \int_{x_t^*} f(\Delta_{a_t}|x_t, \mathcal{P}_{a_t}, a_t, \Theta_i)f(x_t|\mathcal{P}_{a_t}, \Theta_i) dx_t. \quad (4.32)$$

(e) Update probabilities of individual grid points

$$f(\Theta_i|\mathcal{P}_{a_{t+1}}) \propto f(\Delta_{a_t}|\mathcal{P}_{a_t}, a_t, \Theta_i)f(\Theta_i|\mathcal{P}_{a_t}). \quad (4.33)$$

2. Compute the approximate outer model of the system

$$f(\Delta_{a_{t+1}}|\mathcal{P}_{a_{t+1}}, a_{t+1}) = \sum_{i \in i^*} f(\Delta_{a_{t+1}}|\mathcal{P}_{a_{t+1}}, a_{t+1}, \Theta_i)f(\Theta_i|\mathcal{P}_{a_{t+1}}). \quad (4.34)$$

Remark(s) 4.3.1.2

1. Essentially, a bank of filters “indexed” by Θ_i is run and probabilities of individual predictors are updated in a standard Bayesian way.
2. The approach relies on feasibility of individual filters and on the low dimension of the grid. Bootstrap type estimator, [56] and see Section 4.3.2, seems to be an efficient way how to reach it. The time variations of the used grid bring, however, additional troubles.

Problem 4.3.1.3 (How to select the grid and exploit results?) *Acceptable dimensionality of the grid is of a vital importance and it is often reached by adopting additional techniques of a “branch-and-bound” type. It is straightforward to remove Θ_i with low probabilities $f(\Theta_i|\mathcal{P}_{a_t})$ but generating of new promising values Θ_i is not easy. Moreover, the exploitation of the filtering results obtained so far has no systematic solution in multi-step control tasks [57].*

4.3.2 Bootstrap type methods

Knowing the posterior pdf or pdfs describing filtration it would be easy to select a low dimensional non-uniform grid sufficient for a good approximation of the whole pdf. This simple observation forms a basis of a whole set approximate filters that use random samples from the current approximate pdfs for generating new approximations and exploitation of new data for data updating [58]. We describe a version of such *bootstrap filtration* methods that exploits interpretation of random samples as a variable grid.

Under natural conditions of decision making (2.36), we search for practically feasible filtration $f_{t|t}(\Theta) \equiv f(\Theta_t = \Theta | \mathcal{P}_{a_{t+1}}, a_{t+1}) = f(\Theta_t = \Theta | \mathcal{P}_{a_{t+1}})$. The internal quantities Θ_t evolve according to the time evolution model (2.35) $f(\Theta_t | \Theta_{t-1}, \mathcal{P}_{a_t}, a_t)$ and reflect in the observation model $f(\Delta_{a_t} | \Theta_t, \mathcal{P}_{a_t}, a_t)$, see Requirement 2.5.1.1. During the filtering also the prediction $f_{t+1|t}(\Theta) \equiv f(\Theta_{t+1} = \Theta | \mathcal{P}_{a_{t+1}}, a_{t+1})$ of Θ_t is computed, see Proposition 2.5.1.3. Let $\Theta_{i;t|t-1}$, $i \in i^* \equiv \{1, \dots, \dot{i} < \infty\}$ denote random independent samples of Θ taken from the approximate pdf $f_{t|t-1}(\Theta)$. We assume that the values $f_{t|t-1}(\Theta_{i;t|t-1})$ are known. Then, data updating (2.39) gives simply values

$$f_{t|t}(\Theta_{i;t|t-1}) \propto f(\Delta_{a_t} | \Theta_{i;t|t-1}, \mathcal{P}_{a_t}, a_t) f_{t|t-1}(\Theta_{i;t|t-1}).$$

We use pairs $\{\Theta_{i;t|t-1}, f_{t|t}(\Theta_{i;t|t-1})\}_{i \in i^*}$ for a rough approximation of $f_{t|t}(\Theta)$, $\Theta \in \Theta^*$. The simplest approximation by empirical pdf is used here

$$f_{t|t}(\Theta) \equiv \sum_{i \in i^*} \delta(\Theta - \Theta_{i;t|t-1}) w_{i;t|t}, \quad w_{i;t|t} \propto f_{t|t}(\Theta_{i;t|t-1}). \quad (4.35)$$

δ is Dirac delta and $\sum_{i \in i^*} w_{i;t|t} = 1$. With the approximation (4.35), the time updating (2.40) is given by the formula

$$f_{t+1|t}(\Theta) = \sum_{i \in i^*} f(\Theta_{t+1} | \Theta_{i;t|t-1}, \mathcal{P}_{a_{t+1}}, a_{t+1}) w_{i;t|t}.$$

The filtering recursion is then closed by generating random independent samples $\Theta_{i;t+1|t}$, $i \in i^*$ from this pdf. The predictive pdf is obtained simply by approximating the pdf $f_{t+1|t}(\Theta)$ by empirical pdf on the latest samples and using the standard prediction formula (2.38). Let us summarise

Algorithm 4.3.2.1 (Bootstrap filtering)

Off-line phase

Generate independent samples $\{\Theta_{i;1|0}\}_{i \in i^*}$ from the prior pdf $f_{1|0}(\Theta) \equiv f(\Theta_1 | \mathcal{P}_{a_1})$ and set time $t = 0$.

On-line phase, do for $t \in t^*$

1. Collect the newest data Δ_{a_t}, a_t .

2. Perform data updating on samples

$$w_{i;t|t} \propto f_{t|t}(\Theta_{i;t|t-1}) \propto f(\Delta_{a_t} | \Theta_{i;t|t-1}, \mathcal{P}_{a_t}, a_t) f_{t|t-1}(\Theta_{i;t|t-1}), \quad i \in i^*. \quad (4.36)$$

3. Perform time updating

$$f_{t+1|t}(\Theta) = \sum_{i \in i^*} f(\Theta_{t+1} = \Theta | \Theta_{i;t|t-1}, \mathcal{P}_{a_{t+1}}, a_{t+1}) w_{i;t|t}. \quad (4.37)$$

4. Generate independent samples $\{\Theta_{i;t+1|t}\}_{i \in i^*}$ from the pdf $f_{t+1|t}(\Theta)$ and evaluate $w_{i;t+1|t} \propto f_{t+1|t}(\Theta_{i;t+1|t})$.

5. Compute (approximate outer) system model

$$f(\Delta_{a_{t+1}} | \mathcal{P}_{a_{t+1}}, a_{t+1}) = \sum_{i \in i^*} f(\Delta_{a_{t+1}} | \Theta_{i;t+1|t}, \mathcal{P}_{a_{t+1}}, a_{t+1}) w_{i;t+1|t}. \quad (4.38)$$

Remark(s) 4.3.2.2

1. *The algorithm is conceptually simple and universal. It needs a large number of samples. Their generating is generally far from being trivial. For a more complex cases it might make the algorithm infeasible. In other cases sophisticated techniques like Gibbs sampler [59] or adaptive generating [48] might be needed.*
2. *A massive parallelism might be used for making the algorithm computationally efficient.*
3. *Less crude approximations of pdfs than the empirical pdfs are desirable and necessary when time evolution is degenerated and some entries of Θ_t are time invariant. Typically, smooth approximations of Dirac delta are used. A narrow Gaussian pdf “sitting” on the sample may be used. This can be interpreted as addition of an external noise to the original samples. Spline-type or mixture approximations might be used in some cases.*
4. *This group of methods is very adaptive, see Section 4.1 It generates approximators locally around measured behaviour.*

Problem 4.3.2.3 (How to make work bootstrap complete?) *A lot has been reached in development of methods of this type. The state is, however, far from being finished. Problems range from sensitivity to initial values, efficient generating of samples, choice of approximating pdfs up to the choice of the number of samples and numerical (parallel) implementation.*

4.3.3 Extended Kalman filtering

Filtering can be exactly solved if both observation and time-evolution models are Gaussian with mean values depending linearly on the unknown Θ_t and with known covariances. All involved pdfs are Gaussian if the prior pdf is. Then, the filtering reduces to an algebraic evolution of two conditional moments, [10, 60] and also Section ???. In this case, the time-evolution model can be characterised by

$$\begin{aligned}\mathcal{E}[\Theta_{t+1}|\Theta_t, \mathcal{P}_{a_{t+1}}, a_{t+1}] &= A(\mathcal{P}_{a_{t+1}}, a_{t+1})\Theta_t + B(\mathcal{P}_{a_{t+1}}, a_{t+1}) \\ \text{cov}[\Theta_{t+1}|\Theta_t, \mathcal{P}_{a_{t+1}}, a_{t+1}] &= R(\mathcal{P}_{a_{t+1}}, a_{t+1}).\end{aligned}\quad (4.39)$$

A is a square matrix, B is a vector and R is positive semidefinite square matrix.

Similarly, the observation model is characterised by

$$\begin{aligned}\mathcal{E}[\Delta_{a_{t+1}}|\Theta_{t+1}, \mathcal{P}_{a_{t+1}}, a_{t+1}] &= C(\mathcal{P}_{a_{t+1}}, a_{t+1})\Theta_{t+1} + D(\mathcal{P}_{a_{t+1}}, a_{t+1}) \\ \text{cov}[\Delta_{a_{t+1}}|\Theta_{t+1}, \mathcal{P}_{a_{t+1}}, a_{t+1}] &= r(\mathcal{P}_{a_{t+1}}, a_{t+1}).\end{aligned}\quad (4.40)$$

C is a matrix, D a vector, and r is a positive semidefinite square matrix.

All arrays can be *known functions of available data and time.*

The *extended Kalman filter* is obtained if we approximate the treated conditional expectation $\mathcal{E}[\Theta_{t+1}|\Theta_t, \mathcal{P}_{a_{t+1}}, a_{t+1}]$ by the affine term of Taylor expansion around the latest point estimate $\hat{\Theta}$ of Θ_t and $\text{cov}[\Theta_{t+1}|\Theta_t, \mathcal{P}_{a_{t+1}}, a_{t+1}]$ by the absolute term at the same point. Through a similar expansion for the observation model we arrive to forms (4.39), (4.40) and use them for an approximate updating of two moments of the considered filtration by using ordinary Kalman filter. These estimates serve for updating the expansion point.

Remark(s) 4.3.3.1

1. Obviously, the local approximation used in extended Kalman filter is a rather crude. Consequently, a careful start up and various modifications based on a detailed analysis are needed in order to prevent divergence of $\{\hat{\Theta}_t\}_{t \in t^*}$ from $\{\Theta_t\}_{t \in t^*}$, [61].
2. The smoother the model is the higher is chance for success.

Problem 4.3.3.2 (How to combine techniques?) Naturally, the extended Kalman filtering can be combined with partitioning approach, bootstrap methods or mixture estimation, see Section 4.2.3. A systematic way is (to our best knowledge) not available.

4.4 Suboptimal design

4.4.1 Receding horizon

The reduction of the design horizon is the most obvious way to a simplified (suboptimal) design. The reduction obtained by planning just one-step-ahead has been popular for a long time. Dynamic decision making, however, means that consequences of a decision are encountered far behind the time moment of its application. Consequently, the decision that is optimal when judged from a shortsighted perspective might be quite bad one from the long-term viewpoint, [9], see Section ??.

This observation has stimulated search for a compromise between the ideal planning over whole horizon of interest and shortsighted, locally optimising decisions. Knowing that the constructed plan is just an approximation, it is reasonable to correct it permanently. After applying the initial decision(s) implied by the temporary design and after acquiring and exploiting a new measured knowledge about the system state, the design should be repeated. This is essence of the design technique called *receding horizon strategy*. Let us describe its algorithm in the case of additive loss function (2.22), recall (2.31), and for a pre-specified value T of the receding horizon $T < \dot{t}$.

Algorithm 4.4.1.1 (Receding-horizon strategy) Repeat for $t = 1, \dots, \dot{t}$,

1. Find the strategy $\mathcal{R}(t:t+T) = (\mathcal{R}_t, \dots, \mathcal{R}_{t+T})$ approximately minimising

$$\mathcal{E} \left[\sum_{\tau=t}^{t+T} z(\Delta(\tau), a(\tau)) | \mathcal{P}_{a_t} \right]$$

2. Apply $a_{t|t-1} \equiv \mathcal{R}_t(\mathcal{P}_{a_t})$ for the available \mathcal{P}_{a_t} .
3. Measure new data Δ_{a_t}, a_t and use extended experience (perform filtration or estimation) for obtaining improved outer model of the system.

4.4.2 Super-cautious strategy

Let us consider the receding horizon strategy applied at time t . Then, a substantial degree of its complexity is caused by the use of predictive pdfs $\{f(\Delta_{a_\tau} | \mathcal{P}_{a_\tau}, a_\tau)\}_{\tau=t}^{t+T}$ obtained through Bayesian estimation. They have the form, see Proposition 2.5.2.1,

$$f(\Delta_{a_\tau} | \mathcal{P}_{a_\tau}, a_\tau) = \int_{\Theta^*} f(\Delta_{a_\tau} | \Theta, \mathcal{P}_{a_\tau}, a_\tau) f(\Theta | \mathcal{P}_{a_\tau}) d\Theta. \quad (4.41)$$

For a relatively short planning horizon T and a reasonably concentrated parameter estimate $f(\Theta|\mathcal{P}_{a_t})$, the predictors (4.41) can be approximated as follows

$$f(\Delta_{a_\tau}|\mathcal{P}_{a_\tau}, a_\tau) \approx \int_{\Theta^*} f(\Delta_{a_\tau}|\Theta, \mathcal{P}_{a_\tau}, a_\tau) f(\Theta|\mathcal{P}_{a_t}) d\Theta, \quad \tau = t, \dots, t+T. \quad (4.42)$$

This approximation relies on the approximate equality

$$f(\Theta|\mathcal{P}_{a_\tau}) \approx f(\Theta|\mathcal{P}_{a_t}), \quad \tau = t, \dots, t+T. \quad (4.43)$$

The receding horizon strategy combined with the approximation (4.42) is called *super-cautious* as it is based on assumption that we learn nothing about unknown parameters during the planning horizon. The uncertainty of the parameter estimates (4.43) projected into the predictors through the formula (4.42) inhibits excessive decision values. It makes the strategy (typically controller) (super)cautious.

4.4.3 Cautious strategy

The assumption (4.43) is often too strong and it is reasonable to weaken it to

$$f(\Theta|\mathcal{P}_{a_\tau}) \approx F(\tau, f(\Theta|\mathcal{P}_{a_t})), \quad \tau = t, \dots, t+T, \quad \Theta \in \Theta^*. \quad (4.44)$$

The mapping $F(\tau, \cdot)$ modifies the estimate $f(\Theta|\mathcal{P}_{a_t})$ to other, more concentrated, pdfs. The mapping F is chosen beforehand and does not use data belonging to ignorance \mathcal{F}_{a_t} . Typically, the generated pdf has the same 1st moment (expectation) as $f(\Theta|\mathcal{P}_{a_t})$ but its covariance $C_{\tau|t}$ decreases with increasing distance $\tau-t$ in a prespecified way. Often, $C_{\tau|t} = b(\tau-t)C_t \equiv b(\tau-t) \times$ covariance computed for $f(\Theta|\mathcal{P}_{a_t})$. The following non-negative scalar functions $b(\kappa)$ serve as representative examples

$$b(\kappa) = 1/\kappa \text{ or } b(\kappa) = \begin{cases} 1 & \text{if } \kappa = 1 \\ 0 & \text{otherwise} \end{cases}. \quad (4.45)$$

4.4.4 Certainty-equivalence strategy

This strategy is the most wide spread one. It replaces unknown parameter in the parameterised model by a current point estimate $\hat{\Theta}_t$ of unknown parameters

$$f(\Delta_{a_\tau}|\mathcal{P}_{a_\tau}, a_\tau) \approx f(\Delta_{a_\tau}|\hat{\Theta}_t, \mathcal{P}_{a_\tau}, a_\tau), \quad \tau = t, \dots, t+T. \quad (4.46)$$

which corresponds to the approximate parameter estimate

$$f(\Theta|\mathcal{P}_{a_\tau}) \approx \delta(\Theta - \hat{\Theta}_t), \quad \tau = t, \dots, t+T. \quad (4.47)$$

where $\delta(\cdot)$ is Dirac delta function, a formal pdf of the measure concentrated on zero value.

4.4.5 Predictive strategies

Up to now, the reduction of complexity of predictors have been reached by simplifying parameter estimates. There is, however, a whole set of strategies that simplify prediction by working only with point predictions of outputs.

Due to a bit unlucky vocabulary they are known as *predictive strategies* [50, 62]. Obviously, such an approximation may be quite rough when there is a non-negligible uncertainty (lack of knowledge or influence of noise). On the other hand, the gained simplification allows the user to respect hard bounds on behaviour. The importance of technologic restrictions implies popularity of such a treatment.

4.4.6 Active strategies

All outlined strategies are *passive*: when planning their decisions they do not care about learning. At the same time, it is known [5, 6] that the optimal strategy is *dual strategy*. It cares both about immediate decisions and learning process in a balanced way. This motivates a use of measures that support learning. The resulting strategies are known as *active strategies*. Two basic ways are used:

- an external stimulating signal is fed into the closed loop (added to optional quantities like inputs or set points),
- a term is added to the loss function that reflects quality of learning even when a passive-type design is applied [63].

Remark(s) 4.4.6.1

1. *The essence of suboptimal design is demonstrated mostly in combination with estimation. Similar techniques are applicable in conjunction with filtering.*
2. *Quality of approximations depends very much on quality of the parameterised model and on parameter estimates $f(\Theta|\mathcal{P}_{a_t}, a_t)$. Thus, the care devoted to the construction of the parameterised model and the prior pdf $f(\Theta)$, see Section 3.1.4, might decide on a success/failure of the decision making. This observation has been at roots of designing of algorithmic and software tools for the prior design of adaptive controllers [64, 65, 66].*
3. *A substantial effort with significant results has been put into analysis of various approximations. Mostly, however, the results are of an asymptotic nature. A little is known about practically significant finite time-properties of such approximators.*
4. *It is fair to say that the “dangerous” certainty equivalent strategies dominate applications. They are simple, understandable and they can be simply extended to the combination of the design with a state estimator.*
5. *For a large $T < \hat{t}$, the receding horizon strategy gives away majority of computations and it is always in danger that the chosen horizon is too short. Thus, it is reasonable to take the strategy found with the estimate $f(\Theta|\mathcal{P}_{a_t})$ as an initial guess of the strategy re-computed for $f(\Theta|\mathcal{P}_{a_{t+1}})$. This idea is known as iterations spread in (learning) time, [9]. It has been found quite useful but at the same time sensitive to the time variations of the estimated parameters.*
6. *Active strategies are rarely used [3] in spite of known, theoretically proved, danger of passive strategies.*

Problem 4.4.6.2 (Is the back of trick a final stage?) *Still, the approximate design can be called more back of tricks than a systematic framework. Naturally, a systematic solution like the equivalence approach is desirable.*

Problem 4.4.6.3 (Is the fully probabilistic strategy active?) *The strategy generated by the fully probabilistic design, see Proposition 2.4.2.4, is randomised. It can be interpreted as a combination of a deterministic feedback with a dither noise. Thus, there is a chance that it is “naturally” active. This hypothesis, however, has to be inspected in detail.*

4.5 Decomposition of decision making

Repeatedly, splitting of the decision making task in a chain of subtasks is declared as the way of converting optimal design to an approximation of the practical optimal design. At the same time, it is known that any violation of the golden decision making rule (3.1) makes the solution to depart from optimality. Lack of the formal tools for the decomposition leave us with empirical rules in this area.

Here give as an example the list describing such decomposition in the case of adaptive control with learning part based on parameter estimation. Each item in the list has been found as a relatively self-containing decision sub-problem. Whenever possible, we give a reference to the relevant support within the text. The design, as any human activity is iterative. Naturally, the majority of iterations should be concentrated in the off line phase in order to minimise expenses related to the commission of the controller.

4.5.1 Off line phase

The following indicative list of subtasks is solved, often with internal iterations, until the potential user is satisfied.

- Formulate the addressed problem, see Chapter 2
 - get the specification of technical control aims,
 - get the specification of the system,
 - get the specification of the available data, system inputs and outputs
 - get the specification of technologic and complexity restrictions
 - collect the knowledge available
- Perform experimental design and collect data, see Section 3.1.1
- Make data pre-processing, see Section 3.1.1
- Select class of parameterised models, see Section 3.1.3
- Quantify prior knowledge, see Section 3.1.4
- Estimate model structure and control period, see Proposition 3.1.4.9
- Estimate forgetting factor, see Propositions 3.1.4.8, 3.1.4.9
- Perform generalised Bayesian estimation based on prior knowledge and available data; the result will be used as the prior and/or alternative pdf in on-line phase, see Algorithm 3.1.4.11
- Select the loss function (typically, weights of quadratic loss or ideal in fully probabilistic design, see Sections 3.1.5, 2.4.2)
 - Do until the results cannot be improved*
 - Select suboptimal control design and its parameters

- Perform prior design of controller and prediction of closed loop (coupling system \leftrightarrow controller) behaviour by transforming all uncertain quantities into to user-defined quality indicators, formal solution provides Proposition 2.3.2.4, generally Monte Carlo evaluation is needed that becomes feasible using stopping rules, see Section 3.4.1
- Compare results with user’s specification; set estimation is useful to it, see Section 3.3.2

4.5.2 On line phase

These decision making subtasks are solved in real time for $t \in t^*$. Here, there is almost no freedom for iterative trial-and-error solutions. Of course, it is wise to store the data collected during the on-line phase and use them for an improved off-line design.

- Collect and pre-process data, see Section 3.1.1
- Generate reference signals
- Estimate with stabilised forgetting, see Section 4.2.2
- Use receding horizon or iterations-spread-in time design, see Section 4.4.1, in an appropriate version, see Section 3.4.1
- Generate input using the designed strategy and measured data
- Check possible discrepancies and make a finer tuning of optional parameters of the design; this supervision can be based on theory of operator control [67], that exploits mixture estimation, see Sections 3.1.3, 4.2.3

Problem 4.5.2.1 (Completion of the design support) *The presented description just outlines the complexity of the overall chain. Obviously, some steps are weakly supported by the presented theory and algorithms. Some of them are not supported at all. In spite of this our experience indicates that the presented theory provides the most complete unified support available. Still even a partial completion of this support represents significant research challenge.*

4.6 Some numerics

The presented theory should lead to algorithms implemented in computers. Thus, their numerical realisation form a substantial part of the decision making as applied science. Naturally, each decision task requires its specific approach. There is, however, common or wide spread problems that are worth to be mentioned independently. This is done here.

4.6.1 Sharp likelihoods

Probability density functions are the basic object we are dealing with. The main numerical problems with them are encountered in parameter estimation and in testing hypothesis. In these cases, they have tendency to converge to very sharp functions close to Dirac delta. It means that we have to work with logarithms of likelihood functions, $\ln[\mathcal{L}(\Theta, \mathcal{P}_{at})]$ and before converting them into posterior pdfs subtract $\max_{\Theta \in \Theta^*} \ln[\mathcal{L}(\Theta, \mathcal{P}_{at})]$. It is possible because the

likelihood is determined uniquely up to a Θ independent factor. It is advantageous as it prevents overflows and the dynamical range of the posterior pdfs causes at most underflows for values for which the posterior pdf is negligible.

4.6.2 Monte Carlo techniques

Monte Carlo (MC) techniques evaluate complex formulae by generating samples with a specific distribution and by forming (typically point) estimators of the results of interest. They rely on laws of large numbers that guarantee convergence of such estimators to the target value. There is a numerous variants and problems that have to be resolve in order to make this idea work efficiently. Here, we can only touch this rich, useful and interesting area.

Generating of samples

For MC techniques samples from a pre-specified distribution have to be generated. We need them, for instance, in bootstrap-based filtering, see Section 4.3.2.

We denote $x \sim f$ if the sample x is generated according to the pdf f .

Majority of software tools, like MATLAB, provide (pseudo)random number generators that provide independent samples uniformly distributed on interval $(0,1)$, its pdf we denote $\mathcal{U}(0,1)$, or normally distributed with zero mean and unit variance, its pdf we denote $\mathcal{N}(0,1)$. Thus, we assume further on that samples of this type are at disposal. The results summarised in the following proposition serve often for solution of the task addressed.

Proposition 4.6.2.1 (Samples obtained from uniform and normal pdfs) 1. Let $f(x)$ be a pdf of a univariate real variable $x \in x^* \equiv (-\infty, \infty)$ and $r \sim \mathcal{U}(0,1)$. The $x_r \sim f$ if it solve the equation

$$r = \int_{-\infty}^{x_r} f(x) dx. \quad (4.48)$$

2. Let the entries of a vector e be independent and each of them $e_i \sim \mathcal{N}(0,1) \Leftrightarrow e \sim \mathcal{N}(0,I)$ where $I =$ unit matrix. Then,

$$g \equiv \mu + Ge \sim \mathcal{N}(\mu, GG') \quad (4.49)$$

for any vector μ and matrix G of appropriate dimensions.

Proof: ad 1. The equation (4.48) has a unique solution as the integral as a function of the upper bound is continuous non-decreasing function covering interval $(0,1]$ due to the non-negativity and normalization of pdf f , see Proposition 2.3.2.2.

Using Proposition 2.3.2.4 with $\alpha = x_r$, $\beta = r$ $f_T(\beta) = \mathcal{U}(0,1) \equiv 1$ on $(0,1)$, we find $|J(\alpha)| = f(\alpha)$

ad 2. Using Proposition 2.3.2.4, it can be verified that the considered affine transformation maps preserves normality. Transformation of moments is implied either by the same Proposition or can be simply verified using basic properties of expectation, see Proposition 2.3.2.5. \diamond

Usefulness of the transformation (4.48) depends heavily on our ability to solve efficiently this equation. A bunch of other techniques has been developed to cope with (predominant) cases to which it cannot be used.

Prediction of achievable quality

This section presents an example of application of Monte Carlo technique for prior prediction of achievable quality of the optimal strategy [64, 65, 68].

The basic scenario is as follows. For a chosen parameterised model, prior pdf $f(\Theta)$ is supposed to be constructed. Let us assume that for each Θ we are able to find the stationary optimal strategy ($\dot{t} \rightarrow \infty$) that is not admissible as it depends on unknown parameter. Without too much loss of generality, we can assume that strategy is determined by a repetitive use of a single rule \mathcal{R}_Θ . With this strategy, the achieved values of the considered stationary loss function become a function, say $Q(\Theta)$ of Θ only. These values represent lower bound on achievable quality. The admissible strategy can achieve this bound only when it successfully learn the unknown parameters.

As we are uncertain about Θ we are uncertain about values of $Q(\Theta)$. This quality indicator, however, has been computed through a very deterministic optimisation. Thus, conceptually we can get pdf of Q by using Proposition 2.3.2.4. Practically, the proposition cannot be used as the mapping $T : \Theta^* \rightarrow Q^*$ is non-linear, many to one and its values are mostly evaluated numerically. Here, Monte Carlo technique can be directly used. Simply, independent samples $\Theta_i \sim f(\Theta)$, $i \in i^*$ are generated and independent samples $Q_i \equiv Q(\Theta_i)$ computed in the mentioned deterministic way. Then, the desired pdf $f(Q)$ is estimated.

The estimation of the pdf $f(Q)$ is simple as Q is scalar variable.

The estimation of the pdf $f(Q)$ is complex as each sample Q result from demanding optimisation. For this reason, all measures have to be taken in order to get results within acceptable time. For instance,

- A parameterised version of $f(Q)$ is estimated [69].
- Sequential stopping rules [48], see Section 3.4.1, are employed.
- Partial characteristics of $f(Q)$, like credibility intervals, see Section 3.3.2, are searched for.

4.6.3 Quadratic forms

Optimal designs with quadratic loss functions and Gaussian models with expected value formed by a linear function of parameters and observed data are often *practically* optimal. Both learning and optimisation reduce to manipulations with positive semidefinite quadratic forms. This fact is behind the predominant use of this so called *LQG design*. Moreover, other decision problems are often approximated by an LQG one. This explains why we present the summary within this general chapter.

Agreement 4.6.3.1 (Quadratic forms) *Quadratic form $\psi'Q\psi$ is the function of a real column vector $\psi \in \psi^*$, specified by a symmetric matrix $Q = Q'$. A quadratic form is said positive semidefinite if it is non-negative for any $\psi \in \psi^*$. It is called positive definite if $\psi'Q\psi > 0$, $\forall \psi \in \psi^* \setminus 0$. The matrix Q is called kernel of the form and the introduced terminology is used also for it. It means that positive (semi)definite kernel specifies positive (semi)definite quadratic form.*

Proposition 4.6.3.2 (Factorised kernels) *The kernel Q is positive definite iff it can be uniquely expressed (factorised) as follows*

$$Q = \underline{L}\underline{L}' = \underline{U}\underline{U}' = \underline{L}\underline{D}\underline{L}' = \underline{U}\underline{D}\underline{U}', \quad \text{where} \quad (4.50)$$

\underline{L} , (\underline{U}) is lower (upper) triangular matrix with positive diagonal entries

L , (U) is lower (upper) triangular matrix with unit diagonal entries

D , \underline{D} are diagonal matrices with positive entries.

Proof: Let the first decomposition exist. Then, $\psi'Q\psi = \sum_i z_i^2 \geq 0$ with $z \equiv L'\psi$. The matrix L is regular and consequently maps non-zero ψ on non-zero z . Thus, the last inequality is sharp for $\psi \neq 0$. The similar arguments apply for the alternative decompositions.

Let Q be positive definite matrix and let us search for the first decomposition. If we write the desirable identity entry-wise exploit the definition of \underline{L} and symmetry of Q , we get formal algorithm for computing \underline{L}

Algorithm 4.6.3.3 (Choleski algorithm)

$$\begin{aligned}
 i \in i^* &= \{1, \dots, i\} \\
 \underline{L}_{ii} &= \sqrt{Q_{ii} - \sum_{k \leq i-1} \underline{L}_{ik}^2} \\
 j &= \{i+1, \dots, i\} \\
 \underline{L}_{ji} &= \frac{Q_{ij} - \sum_{k \leq i-1} \underline{L}_{ik} \underline{L}_{jk}}{\underline{L}_{ii}} \\
 &\text{end } j \\
 &\text{end } i
 \end{aligned}$$

The direct inspection shows that all quantities on right-hand side are available in time if we understand sums over empty set as zero. It remains to judge whether square-root and division make sense. If we, however, assume that it is not true at an i th step then we get the contradiction with positive definiteness by trying the vector ψ which is zero except of the unity on the i th position.

The LD factorisation is obtained from the Choleski ones by extracting diagonal entries from \underline{L} . Similar considerations verify the factorisations with upper triangular matrices. \diamond

Remark(s) 4.6.3.4

1. The expressions $\underline{L}\underline{L}' = \underline{U}\underline{U}'$ are called Choleski factorisations as the name of the algorithm indicate.
2. Algorithms providing LD or UD factorisations can be derived in the same way. They are even computationally simpler as they do not need to compute square roots. It gives them alternative name square-root free factorisations.
3. LD (UD) factorisations are easily extended (not uniquely) to positive definite Q : whenever $D_{ii} = 0$, ($\underline{D}_{ii} = 0$) the corresponding non-diagonal entries in i th column (row) of L , (U) are set to zero.
4. The factorisation $Q = AA'$ with a general A is not unique even for positive definite Q . It is sufficient to consider any orthogonal matrix T for which $TT' = I = \text{unit matrix}$. Then, obviously, also AT is a factor of Q if A is.

Proposition 4.6.3.5 (Minimisation with factorised kernels) Let $\psi' = [u', x']$ and $Q = LDL'$ with

$$L = \begin{bmatrix} L_u & 0 \\ L_{xu} & L_x \end{bmatrix}, \quad D = \begin{bmatrix} D_u & 0 \\ & D_x \end{bmatrix}$$

where dimensions of L_u, D_u are compatible with the dimension of u . Then,

$$u = -(L'_u)^{-1}L'_{xu}x = \arg \min_{u \in u^*} \psi' LDL' \psi'. \quad (4.51)$$

The reached minimum is

$$\min_{u \in u^*} \psi' LDL' \psi' = x' L_x D_x L'_x x' \quad (4.52)$$

Proof: The minimised quadratic form can be written as a sum of squares of two weighted quadratic norms

$$\psi' LDL' \psi = \|u' L_u + x' L_{xu}\|_{D_u}^2 + \|x' L_x\|_{D_x}^2.$$

Only the first one depends on the optional u . It reaches minimum for the zero argument which gives the unique minimiser. The second one gives the minimum reached. \diamond

Remark(s) 4.6.3.6

1. The minimisation of a quadratic form is very simple when the kernel is a factorised version. More importantly, it is numerically safe as the work with factors guarantees that the minimised form is always at least positive semidefinite even if L is evaluated with errors. The original Q lacks this property.
2. The advantageous numerical properties of factorised versions might be preserved when we do not need to create the complete kernel during manipulations encountered. We indicate the basic algorithm with such a property in case when two kernels should be summed.

Let $Q = LDL'$ and l_i, d_i i th column of L and i th diagonal entry of D_i respectively. Then, $Q = \sum_{i \in i^*} l_i d_i l'_i$ where $l'_i = \underbrace{[0, \dots, 0]}_{(i-1)}, 1, x, \dots, x$ where x denotes entries with arbitrary

values. If the similar notation is used for another kernel $\tilde{Q} = \tilde{L}Q\tilde{L}'$ we have

$$Q + \tilde{Q} = \sum_{i \in i^*} (l_i d_i l'_i + \tilde{l}_i \tilde{d}_i \tilde{l}'_i) \quad (4.53)$$

It is straightforward to derive an algorithm for dyadic reduction recomputing a sum of two weighted dyads with a common leading unit into sum of two dyads where the second gets zero of the leading unity. Taking into account (4.53), it is sufficient to take the first dyad from both decompositions, to reduce one of by the dyadic reduction. Then the reduction is called twice on the triple of dyads having unit on the second position, etc. In this way, LD decomposition of the sum of kernels is directly obtained from factor of summed terms.

The factorised version of *precision matrix*, i.e. inversion of of covariance matrix of a vector ψ havin normal pdf allows us to evaluate simply conditional and marginal pds of its subvectors.

Proposition 4.6.3.7 (Conditioning and marginalisation fo normal pdfs) *Let $\psi' = [u', x']$ be normally distributed with zero mean and $Q \equiv [\text{cov}(\psi)]^{-1} = LDL'$ with*

$$L = \begin{bmatrix} L_u & 0 \\ L_{xu} & L_x \end{bmatrix}, \quad D = \begin{bmatrix} D_u & 0 \\ 0 & D_x \end{bmatrix}$$

where dimensions of L_u, D_u are compatible with the dimension of u . Then, $f(u|x)$ is normal with conditional mean and precision matrix

$$\mathcal{E}[u|x] = -(L'_u)^{-1}L'_{xu}x, \quad [\text{cov}(u|x)]^{-1} = L_uD_uL'_u. \quad (4.54)$$

The marginal pdf $f(x)$ of x is zero mean with precision matrix

$$[\text{cov}(x)]^{-1} = L_xD_xL'_x \quad (4.55)$$

Proof: It is obvious by writing the quadratic form in exponent of multivariate normal form exactly as in proof of Proposition 4.6.3.5 and recall chain rule for pdfs $f(u, x) = f(u|x)f(x)$. \diamond

Remark(s) 4.6.3.8

1. Notice that the position of the variable with respect to which minimisation or conditioning is performed is important and dependent on the factorisation used. This simple observation is often overlook

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